Scholarly Research Exchange

SRX Physics • Volume 2010 • Article ID 592051 • doi:10.3814/2010/592051

Research Article

Ion Plasma Responses to External Electromagnetic Fields

H. W. L. Naus

TNO Defence, Security and Safety, BU Observation Systems, P. O. Box 96864, 2509 JG The Hague, The Netherlands

Correspondence should be addressed to H. W. L. Naus, rik.naus@tno.nl

Received 20 August 2009; Accepted 30 September 2009

The response of ion plasmas to external radiation fields is investigated in a quantum mechanical formalism. We focus on the total electric field within the plasma. For general bandpass signals three frequency regions can be distinguished in terms of the plasma frequency. For low frequencies, the external field is shielded. For high frequencies, the field is not modified. Resonant behavior of the plasma appears for frequencies near the plasma frequency: large internal electric fields and induced currents are present. These effects may be relevant for biological systems. The model is therefore extended to a two-species plasma and additional interactions are studied. The response is not essentially altered. To make the models more realistic, a so-called bath is included. In the weak coupling approximation the resonance frequency is shifted and some damping occurs. Finite temperature effects on the electric field are absent. The energy of the system, however, depends on temperature.

1. Introduction

Ion plasmas subject to external electromagnetic fields are studied in this paper within the theoretical framework of [1]. The latter work focusses on the global residual symmetry of electrodynamics, the displacement symmetry, which has been shown to be realized in the Wigner-Weyl mode in the electron plasma. The so-called zero-modes of the gauge field, that is, *zero-momentum* "photons", play a crucial role. These dynamical quantum mechanical variables describe spatially constant, time-dependent electric fields with three polarization states. The concomitant symmetry is spontaneously broken for free or weakly interacting electromagnetic fields; this explains the vanishing mass of photons, in the sense that they can be interpreted as Goldstone bosons.

In a plasma, however, the interactions between center of mass and zero-modes causes the symmetry to be realized in the Wigner-Weyl mode. This is explicitly demonstrated for a plasma of nonrelativistic electrons using periodic boundary conditions and assuming some uniformly distributed background charge. Its response to an external homogeneous electric field is also addressed in [1].

Here we will apply this formalism to an ion plasma and extend it to the two-species plasma. Such a plasma serves as a model for the "free" ions, for example, sodium, pottassium, calcium, and chloride, present in the intra- and extracellular fluids of the human body. In particular, we want to investigate the plasma response to external electromagnetic fields with various waveforms. If such a response is nontrivial, it may indicate a mechanism for a possible biological effect of radio frequency fields used in, for example, communication. Such mechanisms are hardly known up to now. Since the plasma frequency is crucial for the response, we estimate these frequencies for the ions in the intra- and extracellular fluids.

Further theoretical developments concern additional interactions. To this end, we use the Heisenberg picture of quantum mechanics. In realistic systems, damping and fluctuations are of course expected. In order to include such effects, we extend the plasma with a surrounding medium—the *bath*. It is shown that temperature does not affect the induced electric field in this extended model whereas it changes the energy.

The possibility of effects of electromagnetic fields in biological systems has been addressed from a theoretical physics perspective in [2]. The following remarks elucidate the connection between this seminal study and our present work. We have found a long-range effect, in the sense that zero-mode oscillations are constant in space, and a collective effect, in the sense that the CM motion of the ions plays a crucial role. The screening of charges by ions is already mentioned in [2]. It is also noted that plasma modes of unattached electrons can be a source of electric vibrations. The paper [2] discusses the interaction of polarization waves with unattached ions as well. There is, of course, a vast amount of literature on biological effects of electromagnetic radiation. Some more recent examples are [3–7]. It is not the purpose of this study to discuss these papers; a recent review, including many references, is given in [8].

The outline of this paper is as follows. First we introduce the theoretical description of the periodic ion plasma and the coupling of an external electromagnetic field, that is, essentially the formalism of [1]. Secondly, the plasma response is calculated for linearly polarized electromagnetic fields with a number of relevant waveforms. In Section 4, the results are extended to general bandpass signals. Estimates for realistic ion plasma frequencies within intra- and extracellular fluids are given next. Section 6 deals with the two-species plasma. Additional interactions are introduced in Section 7. The time-dependent energy of the plasma coupled to an external field is calculated as well. Then a surrounding medium, the bath, is included in the plasma model; Section 8 also considers temperature effects. Finally, we present some conclusions, further discussion, and an outlook.

2. Ion Plasma

For the complete description of the periodic plasma we refer to [1]. Note that the original derivation deals with electrons, implying that we need to change the relevant physical parameters like mass, charge and density. To neutralize the charge of the ions, a uniformly distributed background charge is assumed—in our application corresponding to other ions.

2.1. Formalism and Plasma Oscillations. In the following we restrict ourselves to the essential part of the dynamics needed to demonstrate the typical effects. The relevant Hamiltonian is given by

$$H_0 = \frac{1}{2Nm}\left(\vec{P} - Ne\vec{A}^0\right)^2 + \frac{1}{2V}\left(\vec{\Pi}^0\right)^2. \tag{1}$$

It describes the center of mass $\vec{X}$ of N ions, conjugate momentum $\vec{P}$, in interaction with the zero-mode of the gauge field $\vec{A}^0$. The conjugate momentum of this zero-mode $\vec{\Pi}^0$ is proportional to a spatially constant electric field $\vec{E}^0 = -\vec{\Pi}^0/V$. The ion mass is denoted by m, its charge by e and $V = L^3$ is the quantization volume; recall that periodic boundary conditions are imposed. Relative motion, the complete radiation field and their couplings are described by their respective Hamiltonians.

The displacement symmetry, a relic of the original gauge symmetry, is characterized by the operators

$$\vec{D}^0 = Ne\vec{X} - \vec{\Pi}^0, \qquad \Omega(\vec{n}) = \exp\left(-i\frac{2\pi}{eL}\vec{D}^0 \cdot \vec{n}\right), \tag{2}$$

where $\vec{n}$ is integer, thereby respecting the boundary conditions. The transformations Ω shift the zero-mode and accordingly the center of mass momentum. They leave the Hamiltonians invariant, in particular,

$$\Omega(\vec{n}) H_0 \Omega^\dagger(\vec{n}) = H_0. \tag{3}$$

This residual symmetry is extensively discussed in [1].

Here we proceed to the known eigenfunctions of (1):

$$H_0 \Psi_{\vec{n},\vec{m}}\left(\vec{X}, \vec{A}^0\right) = \mathcal{E}_{\vec{m}} \Psi_{\vec{n},\vec{m}}\left(\vec{X}, \vec{A}^0\right) \tag{4}$$

which are given by

$$\Psi_{\vec{n},\vec{m}}\left(\vec{X}, \vec{A}^0\right) = \frac{1}{\sqrt{V}} e^{i(2\pi/L)\vec{n}\cdot\vec{X}} \psi_{\vec{m}}\left(\vec{A}^0 - \frac{2\pi}{NeL}\vec{n}\right). \tag{5}$$

The nonstandard notation for the energy $\mathcal{E}$ is chosen in order to avoid confusion with electric fields. The functions $\psi_{\vec{m}}$ are harmonic oscillator wave functions; the ground state reads

$$\psi_{\vec{0}}(\vec{a}) = \left(\frac{V\omega_p}{\pi}\right)^{3/4} \exp\left(-\frac{1}{2}V\omega_p \vec{a}^2\right), \tag{6}$$

with plasma frequency

$$\omega_p^2 = \frac{Ne^2}{Vm}. \tag{7}$$

The energy eigenvalues depend on the principal quantum number m_p but are independent of center of mass motion, that is, infinitely degenerated:

$$\mathcal{E}_{\vec{m}} = \omega_p\left(\frac{3}{2} + m_p\right); \tag{8}$$

we have taken $\hbar = 1$. The equilibrium position of the harmonic oscillator, however, is determined by the center of mass momentum. The eigenfunctions $\Psi_{\vec{n},\vec{m}}$ are center of mass momentum eigenstates as well. They are no eigenfunctions of the displacement operator $\vec{D}^0$. These are obtained by the linear combinations

$$\Psi_{\vec{X}_0,\vec{n},\vec{m}}\left(\vec{X}, \vec{A}^0\right) = \sum_{\vec{k}} \frac{1}{V} e^{i(2\pi/L)\left(\vec{n}+N\vec{k}\right)\cdot(\vec{X}-\vec{X}_0)} \times \psi_{\vec{m}}\left(\vec{A}^0 - \frac{2\pi}{NeL}\left(\vec{n} + N\vec{k}\right)\right), \tag{9}$$

which are also eigenfunctions of the Hamiltonian, degenerated with (5). Once more, we refer to [1] for further discussion. It is only noted that the expectation value of the current operator $\vec{J} = (1/2Nm)\{\vec{P} - Ne\vec{A}^0, \delta(\vec{x} - \vec{X})\}$ vanishes for center of momentum eigenstates as well as for the gauge invariant states:

$$\begin{aligned} \vec{J}_{\vec{n},\vec{m}}(\vec{x}) &= \left\langle \Psi_{\vec{n},\vec{m}} \middle| \vec{J} \middle| \Psi_{\vec{n},\vec{m}} \right\rangle = 0, \\ \vec{J}_{\vec{X}_0',\vec{X}_0,\vec{n},\vec{m}}(\vec{x}) &= \left\langle \Psi_{\vec{X}_0',\vec{n},\vec{m}} \middle| \vec{J} \middle| \Psi_{\vec{X}_0,\vec{n},\vec{m}} \right\rangle = 0. \end{aligned} \tag{10}$$

2.2. Plasma in Homogeneous Electric Field. In order to assess the effects of pulsed electromagnetic fields in biological systems, we extend the work of [1] on a plasma in an external field. It is assumed that the electric field is spatially constant—this appears to be a good approximation as long as the involved wavelengths are not too small. An arbitrary time dependence $\vec{E}(t)$ is allowed. Such a field couples to the center of mass of ions and the Hamiltonian becomes time dependent:

$$H_0(t) = \frac{1}{2Nm}\left(\vec{P} - Ne\left(\vec{A}^0 + \vec{A}(t)\right)\right)^2 + \frac{1}{2V}\left(\vec{\Pi}^0\right)^2, \quad (11)$$

where the electric field follows from the vector potential:

$$\vec{E}(t) = -\frac{d}{dt}\vec{A}(t). \quad (12)$$

The time-dependent Schrödinger equation is supplemented with the initial condition:

$$\Psi_{\vec{n},\vec{m}}\left(\vec{X}, \vec{A}^0, t = 0\right) = \Psi_{\vec{n},\vec{m}}\left(\vec{X}, \vec{A}^0\right); \quad (13)$$

compare (5). Therefore it is solved with the *ansatz*

$$\begin{aligned}\Psi_{\vec{n},\vec{m}}\left(\vec{X}, \vec{A}^0, t\right) = {} & \frac{1}{\sqrt{V}} e^{-i\varphi(t)} e^{i\vec{k}_{\vec{n}}\cdot\vec{X}} e^{i\vec{A}^0\cdot\vec{d}(t)} \\ & \times \psi_{\vec{m}}\left(\vec{A}^0 - \frac{1}{Ne}\vec{k}_{\vec{n}} - \vec{a}(t)\right)\end{aligned} \quad (14)$$

with $\vec{k}_{\vec{n}} = (2\pi/L)\vec{n}$. Matching the various $\vec{A}^0$ dependences yields the explicit solution for $\vec{a}$ [1]. This is sufficient to calculate the expectation values of the total electric field in the plasma and the current density. Modifying the notation of [1], we explicitly obtain

$$\vec{E}_{\mathrm{tot}}(t) = \vec{E}(t) - \frac{1}{V}\left\langle\vec{\Pi}^0\right\rangle = \vec{E}(t) + \omega_p \,\mathrm{Im}\,\vec{Z}\left[\vec{E}\right], \quad (15)$$

$$\vec{J}(t) = \frac{e}{Vm}\,\mathrm{Re}\,\vec{Z}\left[\vec{E}\right], \quad (16)$$

where

$$\vec{Z}\left[\vec{E}\right] = \int_0^t d\tau e^{i\omega_p(\tau - t)}\vec{E}(\tau). \quad (17)$$

The external field thus induces a current in the plasma; note that the electromagnetic current density follows as $\vec{J}_{\mathrm{em}} = eN\vec{J}$. Concomitantly, the external electric field gets modified. In Section 3, this will be studied in more detail for various wave forms. The initial condition may be altered to the gauge invariant wave function (6). It can be checked that the obtained expectation values do not change.

The obtained expectation values satisfy $\vec{J}_{\mathrm{em}} = -d\vec{E}^0/dt$, consistent with the remaining nontrivial Maxwell equation in the zero-mode sector. Finally, we mention a simple classical model for electron plasma oscillations [9]. If an external electric field is coupled to the electron gas, the model yields similar results for the classical field and current.

3. Fields and Current in Ion Plasma

In this section, we take a linearly polarized external electric field $\vec{E} = E(t)\vec{e}$. The results are then obviously proportional to the constant unit vector $\vec{e}$, $\vec{E}_{\mathrm{tot}} = E_{\mathrm{tot}}(t)\vec{e}$, $\vec{J} = J(t)\vec{e}$, and will therefore be expressed in the amplitudes $E_{\mathrm{tot}}(t)$ and $J(t)$.

3.1. Constant and Harmonic Fields. As in [1] we first consider two simple cases, which enable an analytical calculation of electric field and current in the ion plasma. A time-independent electric field $E(t) = E_0$ yields a harmonic total field:

$$E_{\mathrm{tot}}(t) = E_0 \cos\omega_p t. \quad (18)$$

The expectation value of current density is given by

$$J(t) = \frac{eE_0}{Vm}\frac{\sin\omega_p t}{\omega_p}. \quad (19)$$

Pure harmonic time dependencies with plasma frequency are obtained. On the average, the external field is shielded.

For a periodic external field $E(t) = E_0 \cos\omega t$ one obtains the total field:

$$E_{\mathrm{tot}}(t) = E_0\left(\frac{\omega_p^2}{\omega_p^2 - \omega^2}\cos\omega_p t - \frac{\omega^2}{\omega_p^2 - \omega^2}\cos\omega t\right), \quad (20)$$

and the current

$$J(t) = \frac{E_0 e}{Vm}\left(\frac{\omega_p}{\omega_p^2 - \omega^2}\sin\omega_p t - \frac{\omega}{\omega_p^2 - \omega^2}\sin\omega t\right). \quad (21)$$

For low frequencies, $\omega \ll \omega_p$, the external field is shielded. The total field is harmonic with frequency ω_p. This is not the case for high frequencies, $\omega \gg \omega_p$: the external field is not modified and the induced current is very small. An interesting resonance mechanism appears in the region $\omega \simeq \omega_p$. This is shown in Figure 1, where external and total field as well as the current are depicted. Note that the total field appears to be amplitude modulated. For small frequency difference $\delta_\omega = |\omega_p - \omega|$ this can indeed be analytically verified. The modulation frequency is given by $(1/2)\delta_\omega$ and an amplitude amplification factor $(\omega_p + \omega)/2\delta_\omega$ is obtained.

3.2. Pulsed Fields. For the study of possible bioelectromagnetic effects, a number of specific waveforms are selected. Pulsed signals are of particular interest; typical values for the parameters are used. We present results for total field and current density in the plasma for the various pulsed signals. The integrations cf. (17) have been computed numerically. Figure 2 shows the response to the Gaussian pulse:

$$E(t) = E_0 \exp\left(-\pi\left(\frac{t}{T}\right)^2\right) \quad (22)$$

with $T = 166\,\mathrm{ps}$, for a plasma frequency $f_p = 3\,\mathrm{GHz}$. The result is equivalent to exciting a harmonic oscillator—in our case the zero-mode gauge field. Next, in Figure 3(a),

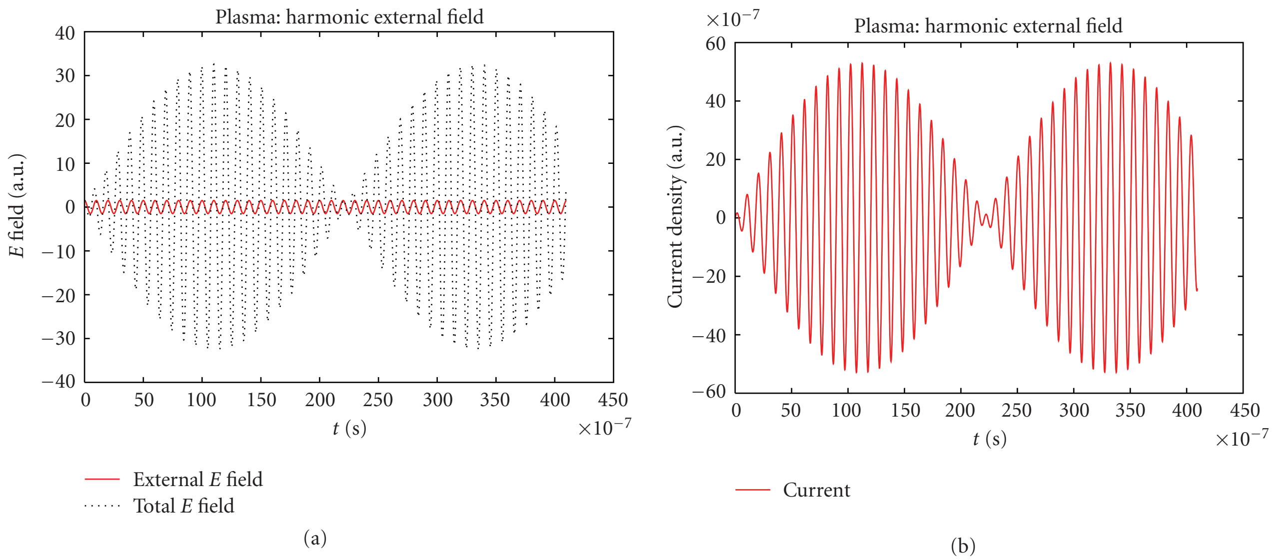


Figure 1: (a) External and total fields. (b) Current density, $f = 0.95$ MHz, $f_p = 1.0$ MHz.

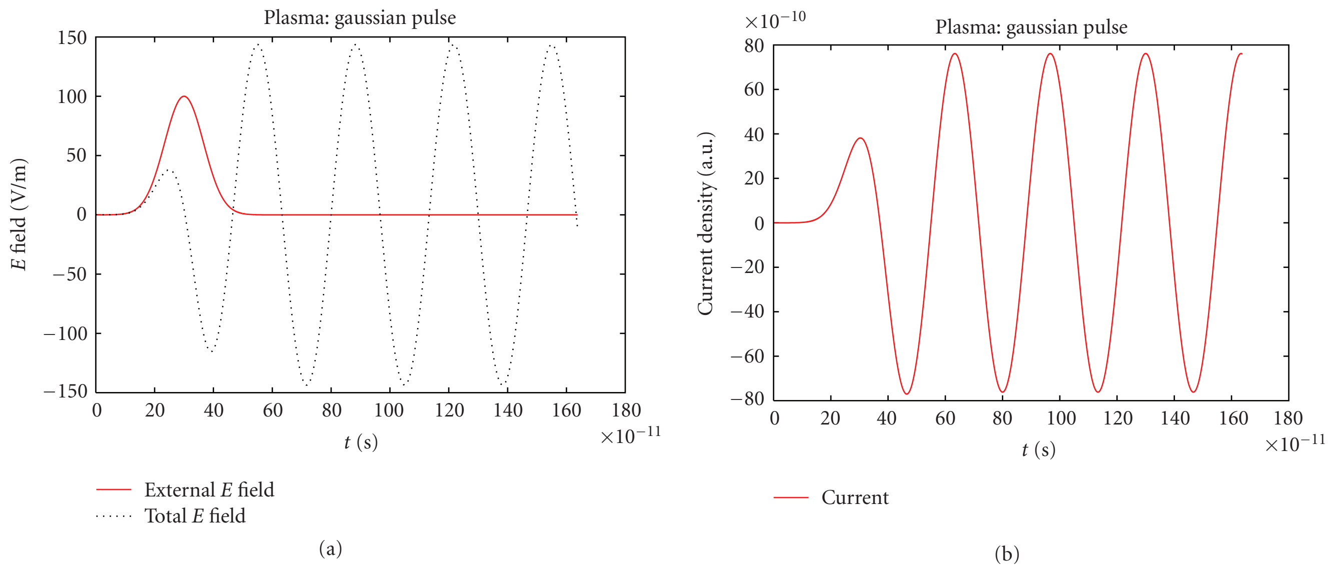


Figure 2: (a) External and total fields, $f_p = 3$ GHz. (b) Current density.

the effects of pulse repetition are shown. The results for the modulated Gaussian pulse

$$E(t) = E_0 \sin \omega t \exp\left(-\pi \left(\frac{t}{T}\right)^2\right), \quad (23)$$

with $T = 332$ ps, $\omega = 4\pi$ GHz, are not essentially different (cf. Figure 4). On the other hand, if the plasma frequency is not in the frequency band of such a pulse one obtains no effects for small ω_p and shielding for large ω_p; see Figure 5. For completeness, we also include amplitude modulation with modulation frequency 0.1 MHz. Shielding and the absence of effects appear in the respective frequency regions. Here we depict the resonance effects for field and current in Figure 6. Figure 7 shows a resonant plasma response for ultra wideband (wideband high-power microwaves) signals:

$$E(t) = E_0 t \exp\left(-\pi \left(\frac{t}{T}\right)^2\right) \quad (24)$$

with $T = 0.2$ ns. Another example of a broadband signal is the nuclear electromagnetic pulse (NEMP)

$$E(t) = E_0 \left(e^{-\alpha t} - e^{-\beta t}\right) : \quad (25)$$

with $\alpha = 4.0 \cdot 10^6\ \mathrm{s}^{-1}$ and $\beta = 4.76 \cdot 10^8\ \mathrm{s}^{-1}$. The resonant plasma response is shown in Figure 8. Note the relatively low plasma frequency used in this NEMP calculation. The results for narrow band high-power microwaves, GSM and chirp

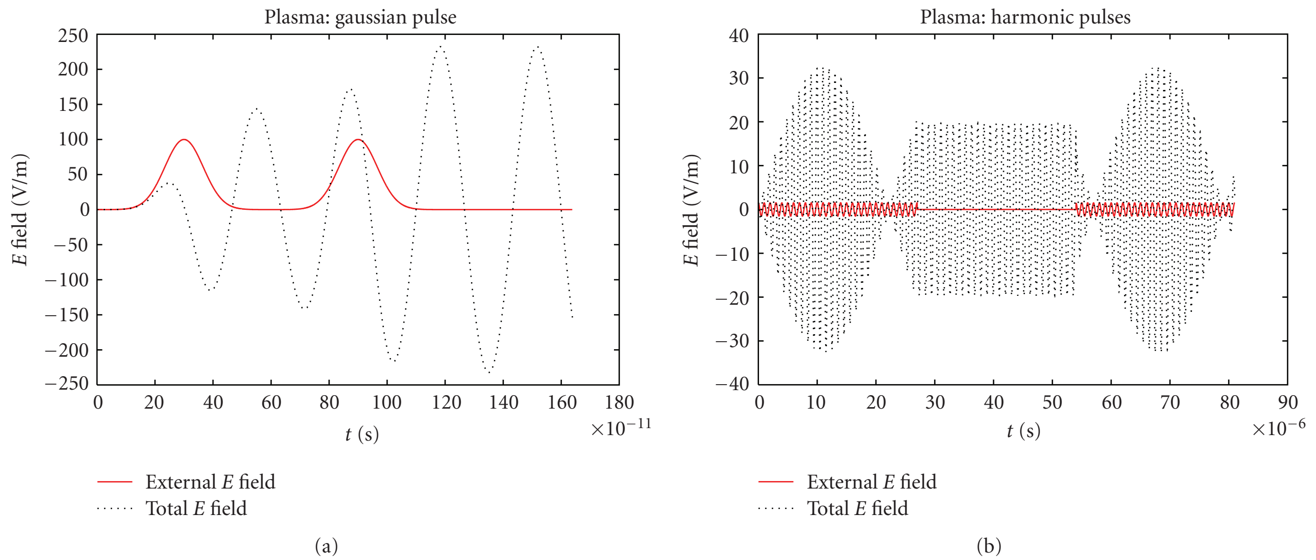


FIGURE 3: External and total fields: (a) Gaussian pulse, repetition; $f_p = 3\,\text{GHz}$; (b) on/off switching, $f_p = 1.0\,\text{MHz}$.

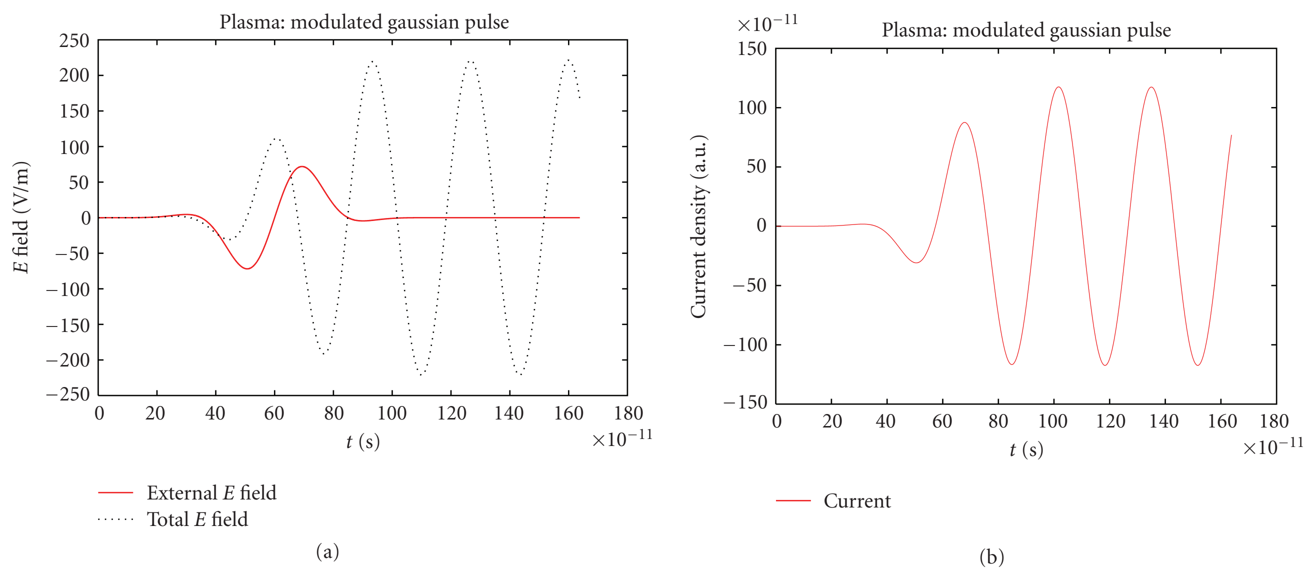


FIGURE 4: (a) External and total fields, $f_p = 3\,\text{GHz}$. (b) Current density.

signals resemble those of a continuous harmonic wave which is switched on and off. The concomitant plasma response, in particular the electric field, is shown in Figure 3(b).

4. General Bandpass Signals

The results obtained so far suggest that there are three regions in the frequency domain, governing the response to general bandpass signals. In order to confirm this, the cosine transform of a real electric field is used [10]:

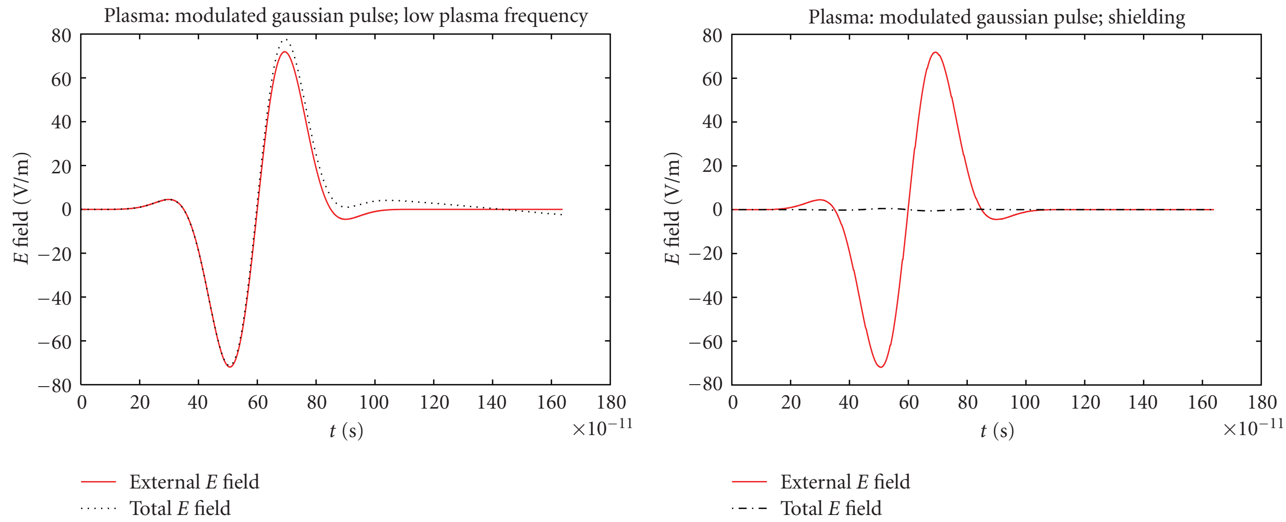
$$E_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty dt E(t) \cos \omega t. \tag{26}$$

The inverse transformation is given by

$$E(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty d\omega E_c(\omega) \cos \omega t. \tag{27}$$

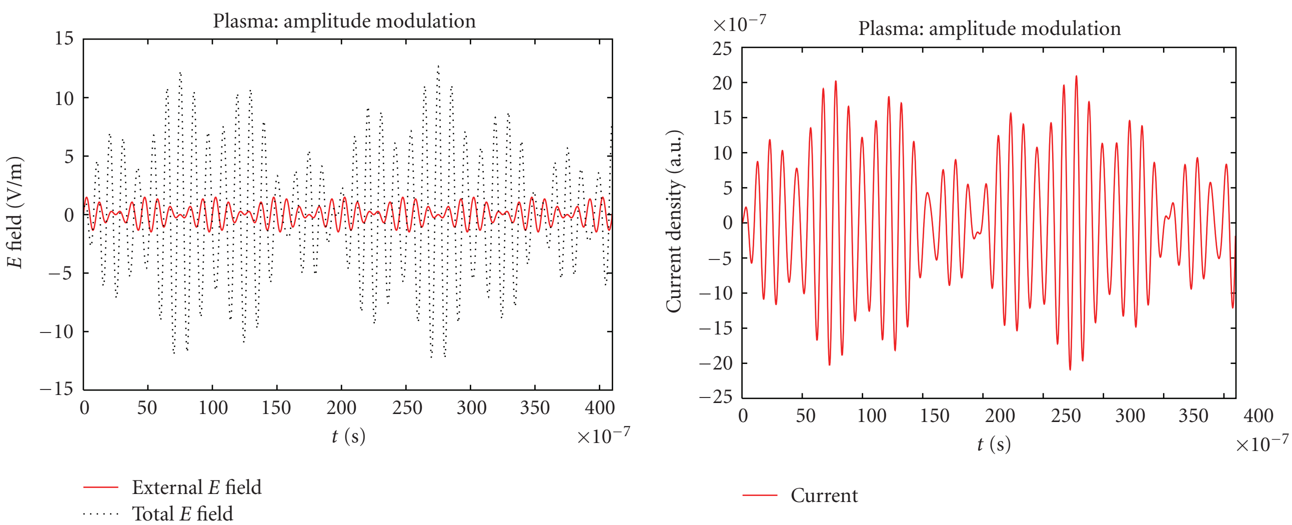
Straightforward integration leads to the following expression for Z:

$$Z(t) = -\frac{1}{2} i \sqrt{\frac{2}{\pi}} \int_0^\infty d\omega E_c(\omega) \times \left\{ \frac{1}{\omega_p + \omega} \left(e^{i\omega t} - e^{-i\omega_p t} \right) + \frac{1}{\omega_p - \omega} \left(e^{-i\omega t} - e^{-i\omega_p t} \right) \right\}. \tag{28}$$



(a) (b)

Figure 5: (a) No effects, $f_p \ll 3$ GHz. (b) Shielding $f_p \gg 3$ GHz.



(a) (b)

Figure 6: (a) External and total fields, $f_p \approx f_c = 1$ MHz. (b) current.

For the total field we obtain

$$E_{\text{tot}}(t) = E(t) - \sqrt{\frac{2}{\pi}} \int_0^\infty d\omega E_c(\omega) \frac{\omega_p^2}{\omega_p^2 - \omega^2} \times \left\{ \cos \omega t - \cos \omega_p t \right\}. \tag{29}$$

This result is generally valid. Let us now consider bandpass signals. This means that the nonvanishing frequency components of the electric field are prominently centered around a carrier frequency ω_c with bandwidth B. It implies that we can approximate the frequency integrals:

$$\int_0^\infty d\omega E_c(\omega) \{\cdots\} \approx \int_{\omega_c - B/2}^{\omega_c + B/2} d\omega E_c(\omega) \{\cdots\}. \tag{30}$$

Now the division into three frequency regions can be made. For $\omega_p \gg \omega_c$ it follows from (29) that the external field is shielded. On the other hand, we see that for $\omega_c \gg \omega_p$ the electric field is almost not influenced by the plasma $E_{\text{tot}}(t) \approx E(t)$. Most interesting is the region $\omega_p \approx \omega_c$, where resonance effects appear

$$E_{\text{tot}}(t) \approx E(t) - \sqrt{\frac{2}{\pi}} \int_{\omega_c - B/2}^{\omega_c + B/2} d\omega E_c(\omega) \frac{\omega_p^2}{\omega_p^2 - \omega^2} \times \left\{ \cos \omega t - \cos \omega_p t \right\}. \tag{31}$$

Equivalent results can be derived by employing the sine-transform which is also defined in [10].

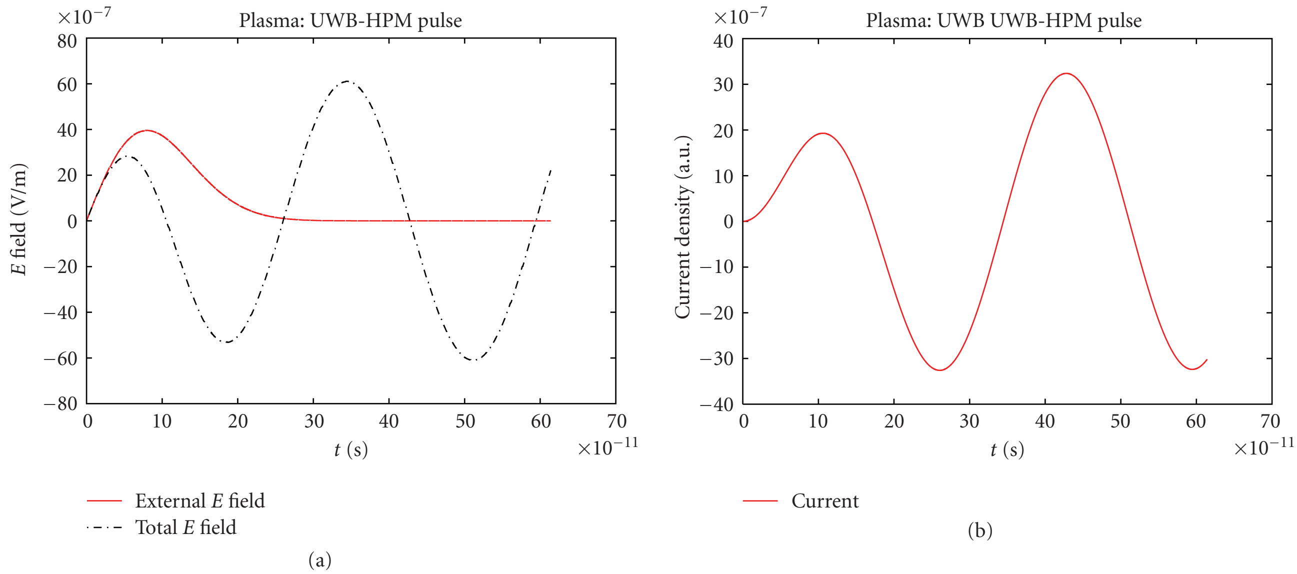


FIGURE 7: (a) External and total fields, $f_p = 3\,\text{GHz}$. (b) Current density.

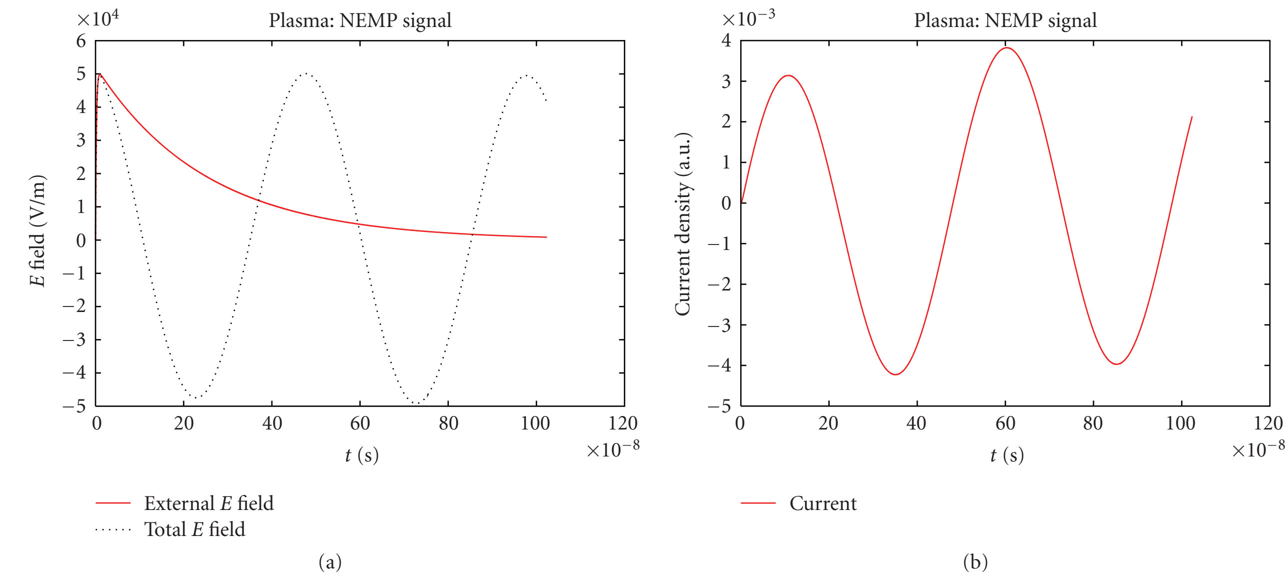


FIGURE 8: (a) External and total fields, $f_p = 20\,\text{MHz}$. (b) Current density

5. Estimates of Plasma Frequencies

As mentioned in the introduction, a possible application of this framework is the effect of electromagnetic fields on biological systems. It is known that intracellular as well as extracellular fluid contains ions, in particular sodium, potassium, calcium, and cloride. Here we estimate the concomitant plasma frequencies for realistic concentrations. The latter is usually given as molar concentration or molarity denoting the number of moles per liter; the symbol for mole/liter is M. In Table 1 typical concentrations are listed. The corresponding plasma frequencies can be computed with (7); a convenient approximate expression is $\omega_p^2 \approx 1.0 \cdot 10^{27} \times q^2(M_C/A)$, with (atomic) mass number A, molar concentration M_C, and the integer charge q in terms of the elementary charge. Thus $q = 2$ for calcium; for the other ions $q = 1$. This leads to the estimated plasma frequencies $f_p = \omega_p/2\pi$ shown in Table 2.

6. Two-Species Plasma

In this section we extend the theory to a plasma with two species of ions. Their charges are eq_1 and $-eq_2$, where e now denotes the elementary charge and q_1, q_2 are positive integers. The respective numbers of particles are denoted by N_1, N_2; we furthermore introduce the charges $Q_1 = eN_1q_1$ and $Q_2 = eN_2q_2$. If $Q_1 = Q_2$, no additional background charge needs to be assumed. Otherwise we still

Table 1: Ion densities.

Ion	Symbol	Atomic and mass number	Intracellular	Extracellular
Sodium	Na^+	$Z = 11,\ A = 23$	10 mM	140 mM
Potassium	K^+	$Z = 19,\ A = 39$	160 mM	4 mM
Calcium	Ca^{2+}	$Z = 20,\ A = 40$	0.1 μM	2 mM
Chloride	Cl^-	$Z = 17,\ A = 35$	5 mM	110 mM

Table 2: Plasma frequencies.

Ion	Symbol	Atomic and mass number	Intracellular	Extracellular
Sodium	Na^+	$Z = 11,\ A = 23$	100 GHz	370 GHz
Potassium	K^+	$Z = 19,\ A = 39$	320 GHz	50 GHz
Calcium	Ca^{2+}	$Z = 20,\ A = 40$	0.5 GHz	70 GHz
Chloride	Cl^-	$Z = 17,\ A = 35$	60 GHz	280 GHz

have to neutralize the net charge by a uniform background distribution [1].

6.1. Hamiltonian and Eigenfunctions. The zero-mode Hamiltonian (cf. (1)) is now given by

$$H_0 = \frac{1}{2V}\left(\Pi^0\right)^2 + \frac{1}{2N_1 m_1}\left(\vec{P}_1 - Q_1\vec{A}^0\right)^2 + \frac{1}{2N_2 m_2}\left(\vec{P}_2 + Q_2\vec{A}^0\right)^2 \tag{32}$$

with particle CM coordinates $\vec{X}_1$, $\vec{X}_2$, conjugate momenta $\vec{P}_1, \vec{P}_2$, and masses m_1 and m_2. The displacement operator is modified as

$$\vec{D}^0 = Q_1\vec{X}_1 - Q_2\vec{X}_2 - \vec{\Pi}^0. \tag{33}$$

It generates the symmetry transformation

$$\Omega(\vec{n}) = \exp\left(-i\frac{2\pi}{eL}\vec{D}^0\cdot\vec{n}\right), \tag{34}$$

which leaves the Hamiltonian invariant. The time-independent Schrödinger equation is solved with the translationally invariant *ansatz*:

$$\Psi\left(\vec{X}_1, \vec{X}_2, \vec{A}^0\right) = \frac{1}{\sqrt{V}}e^{i\vec{p}_1\cdot\vec{X}_1}\frac{1}{\sqrt{V}}e^{i\vec{p}_2\cdot\vec{X}_2}\psi\left(\vec{A}^0\right) \tag{35}$$

with $\vec{p}_1 = (2\pi/L)\vec{n}_1$ and $\vec{p}_2 = (2\pi/L)\vec{n}_2$. Again we obtain harmonic oscillator wave functions $\psi_{\vec{l}}$:

$$\psi\left(\vec{A}^0\right) = \psi_{\vec{l}}\left(\vec{A}^0 - \frac{Q_1 M_2\vec{p}_1 - Q_2 M_1\vec{p}_2}{M_2 Q_1^2 + M_1 Q_2^2}\right), \tag{36}$$

and plasma frequency

$$\omega_p^2 = \frac{M_2 Q_1^2 + M_1 Q_2^2}{M_1 M_2 V}. \tag{37}$$

We have introduced the masses $M_1 = N_1 m_1$ and $M_2 = N_2 m_2$. The energy eigenvalues follow as

$$\mathcal{E}_{\vec{l},\vec{n}_1,\vec{n}_2} = \omega_p\left(\frac{3}{2} + l_p\right) + \frac{\left(Q_2\vec{p}_1 + Q_1\vec{p}_2\right)^2}{2\left(M_2 Q_1^2 + M_1 Q_2^2\right)}. \tag{38}$$

In contrast to the one-ion plasma, there are nonvanishing contributions of the center of mass momenta. Concomitantly, the eigenstates support nonvanishing expectation values of the current density $\vec{J}_{\vec{n}_1,\vec{n}_2,\vec{l}}(\vec{x})$

$$\begin{aligned}\Big\langle \Psi_{\vec{n}_1,\vec{n}_2,\vec{l}} \Big| &\frac{1}{2N_1 m_1}\left\{\vec{P}_1 - Q_1\vec{A}^0, \delta\left(\vec{x} - \vec{X}_1\right)\right\} \\ &+ \frac{1}{2N_2 m_2}\left\{\vec{P}_2 + Q_2\vec{A}^0, \delta\left(\vec{x} - \vec{X}_2\right)\right\}\Big| \Psi_{\vec{n}_1,\vec{n}_2,\vec{l}}\Big\rangle \\ &= \frac{1}{V}\frac{Q_1 + Q_2}{M_2 Q_1^2 + M_1 Q_2^2}\left(Q_2\vec{p}_1 + Q_1\vec{p}_2\right).\end{aligned} \tag{39}$$

The constructed eigenfunctions transform under displacements as

$$\Omega\left(\vec{k}\right)\Psi_{\vec{n}_1,\vec{n}_2,\vec{l}} = \Psi_{\vec{n}_1 - Q_1\vec{k},\vec{n}_2 + Q_2\vec{k},\vec{l}}. \tag{40}$$

Just like in the one-component plasma there is an infinite degeneracy:

$$\begin{aligned} &H_0\Psi_{\vec{n}_1 + Q_1\vec{k},\vec{n}_2 - Q_2\vec{k},\vec{l}}\left(\vec{X}_1 - \vec{X}_{01}, \vec{X}_2 - \vec{X}_{02}, \vec{A}^0\right) \\ &= \mathcal{E}_{\vec{l},\vec{n}_1 + Q_1\vec{k},\vec{n}_2 - Q_2\vec{k}}\Psi_{\vec{n}_1 + Q_1\vec{k},\vec{n}_2 - Q_2\vec{k},\vec{l}}\left(\vec{X}_1 - \vec{X}_{01}, \vec{X}_2 - \vec{X}_{02}, \vec{A}^0\right) \\ &= \mathcal{E}_{\vec{l},\vec{n}_1,\vec{n}_2}\Psi_{\vec{n}_1 + Q_1\vec{k},\vec{n}_2 - Q_2\vec{k},\vec{l}}\left(\vec{X}_1 - \vec{X}_{01}, \vec{X}_2 - \vec{X}_{02}, \vec{A}^0\right).\end{aligned} \tag{41}$$

This enables the construction of gauge invariant eigenstates:

$$\Psi^G_{\vec{X}_{01},\vec{X}_{02},\vec{n}_1,\vec{n}_2,\vec{l}} = \sum_{\vec{k}}\Psi_{\vec{n}_1 + Q_1\vec{k},\vec{n}_2 - Q_2\vec{k},\vec{l}}\left(\vec{X}_1 - \vec{X}_{01}, \vec{X}_2 - \vec{X}_{02}, \vec{A}^0\right). \tag{42}$$

Indeed one can verify that

$$\Omega\left(\vec{j}\right)\Psi^G_{\vec{X}_{01},\vec{X}_{02},\vec{n}_1,\vec{n}_2,\vec{l}} = e^{-i(2\pi/L)\vec{j}\cdot(Q_1\vec{X}_{01} - Q_2\vec{X}_{02})} \times \Psi^G_{\vec{X}_{01},\vec{X}_{02},\vec{n}_1,\vec{n}_2,\vec{l}}. \tag{43}$$

6.2. External Homogeneous Field. Again we study the coupling of a spatially constant electric field with arbitrary time dependence (12). The time-dependent Hamiltonian follows as

$$H_0(t) = \frac{1}{2V}\left(\vec{\Pi}^0\right)^2 + \frac{1}{2M_1}\left(\vec{P}_1 - Q_1(\vec{A}^0 + \vec{A}(t))\right)^2 + \frac{1}{2M_2}\left(\vec{P}_2 + Q_2(\vec{A}^0 + \vec{A}(t))\right)^2. \quad (44)$$

The initial condition is taken as

$$\Psi_{\vec{n}_1,\vec{n}_2,\vec{l}}\left(\vec{X}_1,\vec{X}_2,\vec{A}^0, t = 0\right) = \Psi_{\vec{n}_1,\vec{n}_2,\vec{l}}\left(\vec{X}_1,\vec{X}_2,\vec{A}^0\right). \quad (45)$$

The time-dependent Schrödinger equation can be solved with an *ansatz* analogous to (14):

$$\Psi_{\vec{n}_1,\vec{n}_1,\vec{l}}\left(\vec{X}_1,\vec{X}_2,\vec{A}^0, t\right) = \frac{1}{V} e^{-i\varphi(t)} e^{i\vec{p}_1(\vec{n}_1)\cdot\vec{X}_1} \times e^{i\vec{p}_2(\vec{n}_2)\cdot\vec{X}_2} e^{i\vec{A}^0\cdot\vec{d}(t)} \psi_{\vec{l}}\left(\vec{A}^0 - \vec{q}_{\vec{n}_1,\vec{n}_2} - \vec{a}(t)\right), \quad (46)$$

where $\vec{q}_{\vec{n}_1,\vec{n}_2} = (M_2 Q_1^2 + M_1 Q_2^2)^{-1}(Q_1 M_2 \vec{p}_1 - Q_2 M_1 \vec{p}_2)$. After some algebra, one verifies that the previous differential equation for $\vec{a}(t)$ is not modified in the two-species case. The differential equation for the phase $\varphi(t)$, however, is changed. For the evaluation of the expectation values of electric field and current one does need the explicit solution of the phase. The resulting current can be written as

$$\vec{J}(t) = \frac{1}{V}\frac{Q_1 + Q_2}{M_2 Q_1^2 + M_1 Q_2^2}\left(Q_2\vec{p}_1 + Q_1\vec{p}_2\right) + \frac{M_1 Q_2 - M_2 Q_1}{V M_1 M_2}\left(\vec{a}(t) + \vec{A}(t)\right), \quad (47)$$

and total field is given by

$$\vec{E}_{\text{tot}}(t) = \vec{E}(t) - \frac{d\vec{a}(t)}{dt}, \quad (48)$$

with (cf. (17) and [1])

$$\vec{a}(t) = \omega_p \int_0^t d\tau \vec{A}(\tau) \sin\omega_p(\tau - t). \quad (49)$$

Thus we obtain that the response of the two-species plasma is not essentially different from the one-ion case. The plasma frequency is changed according to (37) and the induced additional current has a different prefactor. It is finally mentioned that one may also impose the initial condition analogous to (45) in terms of gauge invariant states (42) without changing these results.

7. Plasma: Additional Interactions

7.1. Heisenberg Picture. Thus far we have studied the ion plasma using the Schrödinger picture of quantum mechanics. From now on, we exploit the Heisenberg picture since it has turned out to be more convenient for our calculations. Earlier results for the plasma are readily reproduced and extended by a calculation of the energy. Then we reinvestigate the two-species plasma within this framework. The plasma model extended with an additional interacting degree of freedom is studied as well; it can be seen as the prelude to include a bath [11]—as will be done in Section 8.

In the Heisenberg picture states in the Hilbert space are time-independent whereas operators $O(t)$ all depend on time. Their equation of motion is governed by the Hamiltonian:

$$\frac{dO(t)}{dt} = i[H, O(t)] + \frac{\partial O(t)}{\partial t}. \quad (50)$$

Expectation values follow as

$$\langle O(t)\rangle = \langle \Psi_0 \mid O(t) \mid \Psi_0\rangle, \quad (51)$$

where Ψ_0 denotes the time-independent state. For the Hamiltonian one of course obtains

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \quad (52)$$

Below we focus on solving the equation of motion for the electric field operator.

7.2. Plasma

7.2.1. Formalism. We start by analyzing the one-component plasma in the Heisenberg picture. The Hamiltonian (cf. (1)) can be rewritten as

$$H_0 = \frac{1}{2}V\omega_p^2\left(\vec{A}^0 - \frac{1}{Ne}\vec{P}\right)^2 + \frac{1}{2V}\left(\vec{\Pi}^0\right)^2. \quad (53)$$

Next we define the creation and annihilation operators for $k = 1, 2, 3$ (x, y, z) as

$$a_k^\dagger = \frac{1}{\sqrt{2V\omega_p}}\left(\Pi_k^0 + iV\omega_p A_k^0\right), \qquad a_k = \frac{1}{\sqrt{2V\omega_p}}\left(\Pi_k^0 - iV\omega_p A_k^0\right). \quad (54)$$

By means of $[\Pi_k^0, A_l^0] = -i\delta_{kl}$ one readily verifies the basic commutation rule $[a_k, a_l^\dagger] = \delta_{kl}$. Expressing the zero-mode vector potential and electric field in the a-operators

$$\Pi_k^0 = \sqrt{\frac{1}{2}V\omega_p}\left(a_k + a_k^\dagger\right), \qquad A_k^0 = \frac{i}{\sqrt{2V\omega_p}}\left(a_k - a_k^\dagger\right) \quad (55)$$

yields for the Hamiltonian

$$H_0 = \omega_p\left(a_k^\dagger a_k + \frac{3}{2}\right) - \frac{i\omega_p}{Ne}\sqrt{\frac{1}{2}V\omega_p}\left(a_k - a_k^\dagger\right)P_k + \frac{1}{2Nm}\vec{P}^2. \quad (56)$$

Note that we use the summation convention. For the CM momentum one gets

$$\frac{d\vec{P}}{dt} = i\left[H_0, \vec{P}\right] = 0. \tag{57}$$

Consequently, the momentum is conserved and we can replace the operator by its eigenvalue $\vec{P} = \vec{k_{\vec{n}}}$. We again impose periodic boundary conditions. At this point one can derive the equations of motion for the creation and annihilation operators and solve them. We proceed, however, by defining the "shifted" operators:

$$\begin{aligned} b_k^\dagger &= a_k^\dagger - i\alpha_k, \\ b_k &= a_k + i\alpha_k \end{aligned} \tag{58}$$

with $\vec{\alpha} = (1/Ne)\sqrt{(V\omega_p/2)}\vec{k_{\vec{n}}}$. Note that we omit the n-dependence of $\vec{\alpha}$ in our notation. The new operators obviously fulfill $[b_k, b_l^\dagger] = \delta_{kl}$. The electric field is given by

$$\Pi_k^0 = \sqrt{\frac{1}{2}V\omega_p}\left(b_k + b_k^\dagger\right). \tag{59}$$

In terms of the shifted operators the Hamiltonian reads

$$H_0 = \omega_p\left(b_k^\dagger b_k + \frac{3}{2}\right). \tag{60}$$

One recognizes the Hamiltonian of a three-dimensional harmonic oscillator. The equation of motion for b_k follows as

$$\frac{db_k}{dt} = i[H_0, b_k] = -i\omega_p b_k \tag{61}$$

with solution

$$b_k(t) = b_k(0)e^{-i\omega_p t}. \tag{62}$$

For the adjoint operator one obtains

$$b_k^\dagger(t) = b_k^\dagger(0)e^{i\omega_p t}. \tag{63}$$

If we choose as state Ψ_0 one of the eigenstates of H_0, of course with the above specified CM momentum $\vec{k_{\vec{n}}}$, then the expectation value of the electric field vanishes.

7.2.2. Plasma in External Electric Field. Once again we couple an external homogeneous electric field (12). The Hamilton operator is modified as

$$H_0(t) = \frac{1}{2}V\omega_p^2\left(\vec{A}^0 + \vec{A}(t) - \frac{1}{Ne}\vec{P}\right)^2 + \frac{1}{2V}\left(\vec{\Pi}^0\right)^2. \tag{64}$$

The CM momentum remains a conserved quantity; thus we insert $\vec{P} = \vec{k_{\vec{n}}}$. Next we introduce the creation and annihilation operators $a_k^\dagger, a_k$ and, subsequently, the shifted operators $b_k^\dagger$ and b_k. Then we arrive at

$$\begin{aligned} H_0(t) = \omega_p\left(b_k^\dagger b_k + \frac{3}{2}\right) + i\omega_p\sqrt{\frac{1}{2}V\omega_p}\left(b_k - b_k^\dagger\right)A_k(t) \\ + \frac{1}{2}V\omega_p^2\vec{A}^2(t). \end{aligned} \tag{65}$$

Thus we get for the time development:

$$\frac{db_k}{dt} = i[H_0, b_k] = -i\omega_p b_k - \omega_p\sqrt{\frac{1}{2}V\omega_p}A_k(t), \tag{66}$$

which can be solved using the method of "variation of constants." In the solution of the homogeneous equation $b_k(t) = b_k(0)\exp(-i\omega_p t)$, we replace $b_k(0)$ by a function of time $\beta_k(t)$. This yields

$$\frac{d\beta_k}{dt} = -\omega_p\sqrt{\frac{1}{2}V\omega_p}e^{i\omega_p t}A_k(t), \tag{67}$$

which can be readily integrated. In this way we obtain

$$\begin{aligned} b_k(t) &= e^{-i\omega_p t}\left(b_k(0) - \omega_p\sqrt{\frac{1}{2}V\omega_p}\int_0^t d\tau e^{i\omega_p\tau}A_k(\tau)\right), \\ b_k^\dagger(t) &= e^{i\omega_p t}\left(b_k^\dagger(0) - \omega_p\sqrt{\frac{1}{2}V\omega_p}\int_0^t d\tau e^{-i\omega_p\tau}A_k(\tau)\right), \end{aligned} \tag{68}$$

which immediately gives

$$\begin{aligned} \Pi_k^0 = &\sqrt{\frac{1}{2}V\omega_p}\left(b_k(0)e^{-i\omega_p t} + b_k^\dagger(0)e^{i\omega_p t}\right) \\ &- V\omega_p^2\int_0^t d\tau\cos\omega_p(t-\tau)A_k(\tau). \end{aligned} \tag{69}$$

Let us choose the state Ψ_0 as eigenstate of H_0. It implies for the expectation value of the total electric field:

$$\begin{aligned} \vec{E}_{\text{tot}}(t) &= \vec{E}(t) - \frac{1}{V}\left\langle\vec{\Pi}^0\right\rangle \\ &= \vec{E}(t) + \omega_p^2\int_0^t d\tau\cos\omega_p(t-\tau)\vec{A}(\tau). \end{aligned} \tag{70}$$

This result is equivalent to (15) and (17) as can be verified by means of integration by parts.

7.2.3. Energy Considerations. The expectation value of the time-dependent Hamiltonian (65) can be conveniently calculated in the Heisenberg picture. The initial state Ψ_0 is taken as above with harmonic oscillator quantum numbers m_1, m_2, m_3. Omitting this calculation redundant CM-label $\vec{n}$, we thus compute the energy $\mathcal{E}(t)$, as

$$\mathcal{E}(t) = \langle H_0(t)\rangle = \langle\Psi_0 \mid H_0(t) \mid \Psi_0\rangle = \langle\psi_{\vec{m}} \mid H_0(t) \mid \psi_{\vec{m}}\rangle. \tag{71}$$

Using the time-developments (68), we obtain

$$\begin{aligned} \mathcal{E}(t) = &\omega_p\left(m_p + \frac{3}{2}\right) + \frac{1}{2}V\omega_p^2 \\ &\times\left(\omega_p^2\vec{Z}^*\left[\vec{A}\right]\cdot\vec{Z}\left[\vec{A}\right] + 2\omega_p\vec{A}(t)\cdot\operatorname{Im}\vec{Z}\left[\vec{A}\right]\right. \\ &\left.+ \vec{A}(t)\cdot\vec{A}(t)\right) \end{aligned} \tag{72}$$

(cf. (17)). Integration by parts yields

$$\omega_p \vec{Z}\left[\vec{A}\right] = -i\left(\vec{A}(t) + \vec{Z}\left[\vec{E}\right]\right). \quad (73)$$

With this relation we can write the energy in terms of the external field as

$$\mathcal{E}(t) = \omega_p\left(m_p + \frac{3}{2}\right) + \frac{1}{2}V\omega_p^2\vec{Z}^*\left[\vec{E}\right]\cdot\vec{Z}\left[\vec{E}\right]. \quad (74)$$

This result can also be expressed in the expectation values of electric field (15) and (electromagnetic) current (16) as

$$\begin{aligned}\mathcal{E}(t) &= \omega_p\left(m_p+\frac{3}{2}\right)+\frac{1}{2}V^2Nm\vec{J}^{\,2}(t)+\frac{1}{2}V\left(\vec{E}_{\text{tot}}(t)-\vec{E}(t)\right)^2\\ &= \omega_p\left(m_p + \frac{3}{2}\right) + \frac{V}{2\omega_p^2}\vec{J}^{\,2}_{\text{em}}(t) + \frac{1}{2}V\vec{E}^0(t)^2.\end{aligned} \quad (75)$$

The time-dependent terms can be interpreted as the classical energy, expressed in classical current and field, induced by the external field. Note that this additional energy, just as current and field, does not depend on the quantum numbers of the initial state.

7.3. Two-Species Plasma. Using the Heisenberg picture, the two-species plasma is reanalyzed. The Hamiltonian (32) is rewritten as

$$\begin{aligned}H_0 = &\frac{1}{2V}\left(\vec{\Pi}^0\right)^2 + \frac{1}{2}\omega_p^2\left(\vec{A}^0\right)^2 - \frac{Q_1}{M_1}\vec{P}_1\cdot\vec{A}^0\\ &+ \frac{Q_2}{M_2}\vec{P}_2\cdot\vec{A}^0 + \frac{1}{2M_1}\vec{P}_1^2 + \frac{1}{2M_2}\vec{P}_2^2,\end{aligned} \quad (76)$$

where the plasma frequency ω_p is given by (37). In terms of the creation and annihilation operators defined in (54), we get

$$\begin{aligned}H_0 = &\omega_p\left(a_k^\dagger a_k + \frac{3}{2}\right) + \frac{1}{2M_1}\vec{P}_1^2 + \frac{1}{2M_2}\vec{P}_2^2\\ &- \frac{i}{\sqrt{2V\omega_p}}\left(\frac{Q_1}{M_1}\left(\vec{P}_1\right)_k - \frac{Q_2}{M_2}\left(\vec{P}_2\right)_k\right)\left(a_k - a_k^\dagger\right).\end{aligned} \quad (77)$$

Both CM operators commute with the Hamiltonian. Therefore, we can replace the operators by their respective eigenvalues, that is, $\vec{p}_1$ and $\vec{p}_2$:

$$H_0 = \omega_p\left(a_k^\dagger a_k + \frac{3}{2}\right) - i\beta_k\left(a_k - a_k^\dagger\right) + \frac{1}{2M_1}\vec{p}_1^{\,2} + \frac{1}{2M_2}\vec{p}_2^{\,2} \quad (78)$$

with $\vec{\beta} = (1/\sqrt{2V\omega_p})((Q_1/M_1)\vec{p}_1 - (Q_2/M_2)\vec{p}_2)$. Introducing the shifted operators

$$\begin{aligned}b_k &= a_k + i\gamma_k,\\ b_k^\dagger &= a_k^\dagger - i\gamma_k,\end{aligned} \quad (79)$$

where $\vec{\gamma} = \vec{\beta}/\omega_p = \sqrt{(1/2)V\omega_p}((Q_1M_2\vec{p}_1 - Q_2M_1\vec{p}_2)/(M_2Q_1^2 + M_1Q_2^2))$, diagonalizes the Hamiltonian

$$H_0 = \omega_p\left(b_k^\dagger b_k + \frac{3}{2}\right) + \frac{\left(Q_2\vec{p}_1 + Q_1\vec{p}_2\right)^2}{2\left(M_2Q_1^2 + M_1Q_2^2\right)}. \quad (80)$$

Its energy eigenvalues obviously agree with those calculated in the Schrödinger picture (38).

The inclusion of a homogeneous external electric field yields additional terms in the Hamiltonian:

$$\begin{aligned}H_0(t) = &H_0 + \frac{1}{2}V\omega_p^2\vec{A}(t)^2 + V\omega_p^2\vec{A}(t)\cdot\vec{A}^0\\ &- \frac{Q_1}{M_1}\vec{P}_1\cdot\vec{A}(t) + \frac{Q_2}{M_2}\vec{P}_2\cdot\vec{A}(t).\end{aligned} \quad (81)$$

The CM momenta also commute with the modified Hamiltonian and therefore we replace them by their eigenvalues $\vec{p}_1$ and $\vec{p}_2$. Next we introduce the creation and annihilation operators $a_k, a_k^\dagger$ and, eventually, b_k and $b_k^\dagger$. Straightforward algebra then leads to

$$\begin{aligned}H_0(t) = &\omega_p\left(b_k^\dagger b_k + \frac{3}{2}\right) + i\omega_p\sqrt{\frac{1}{2}V\omega_p}\left(b_k - b_k^\dagger\right)A_k(t)\\ &+ \frac{1}{2}V\omega_p^2\vec{A}^2(t),\end{aligned} \quad (82)$$

which is formally identical to (65)—except for the fact that the plasma frequency is altered. Note that the plasma frequency appears also in the definition of the creation and annihilation operators. Herewith our previous results for the two-species plasma are confirmed.

7.4. Plasma: Additional Interaction and External Field. In this section, we include an additional degree of freedom which interacts with the zero-mode in the one-component plasma model. The interaction is in principle taken from [11] where a bath is coupled to a system, as will be studied below as well. Since we want to respect the displacement symmetry, however, a modification is necessary. We also couple the external homogeneous electric field from the onset. The complete Hamiltonian is thus taken as

$$H(t) = H_0(t) + H_b + H_I(t), \quad (83)$$

where $H_0(t)$ is given by (64) and H_b represents a simple harmonic oscillator:

$$H_b = \frac{1}{2}\vec{p}^{\,2} + \frac{1}{2}\omega_1^2\vec{q}^{\,2}. \quad (84)$$

The original interaction term [11] reads

$$H_I^0 = \frac{g}{\sqrt{\omega_p\omega_1V}}\left(\vec{p}\cdot\vec{\Pi}^0 + V\omega_p\omega_1\vec{q}\cdot\vec{A}^0\right) \quad (85)$$

with coupling constant g. It has been actually been introduced as a simpler expression in terms of creation and annihilation operators. This term, however, explicitly breaks the displacement symmetry. The corresponding symmetric (gauge invariant) interaction is given by

$$H_I = \frac{g}{\sqrt{\omega_p \omega_1 V}} \left(\vec{p} \cdot \vec{\Pi}^0 + V \omega_p \omega_1 \vec{q} \cdot \left(\vec{A}^0 - \frac{1}{Ne} \vec{P} \right) \right). \quad (86)$$

Thus the additional degree of freedom couples not only to the zero-mode but also to the CM of the ions. Coupling the external electric fields finally yields

$$H_I(t) = \frac{g}{\sqrt{\omega_p \omega_1 V}} \times \left(\vec{p} \cdot \vec{\Pi}^0 + V \omega_p \omega_1 \vec{q} \cdot \left(\vec{A}^0 + \vec{A}(t) - \frac{1}{Ne} \vec{P} \right) \right). \quad (87)$$

We proceed as above, that is we introduce creation and annihilation operators for the zero-mode, substitute for the CM momentum its eigenvalue $\vec{k_{\vec{n}}}$ and replace the $a_k, a_k^\dagger$ by their shifted operators b_k and $b_k^\dagger$. Moreover, for the additional degree of freedom we define

$$c_k = \frac{1}{\sqrt{2\omega_1}} (p_k - i\omega_1 q_k), \qquad c_k^\dagger = \frac{1}{\sqrt{2\omega_1}} (p_k + i\omega_1 q_k), \quad (88)$$

which obey $[c_k, c_l^\dagger] = \delta_{kl}$. After some computations one obtains for the Hamiltonian

$$\begin{aligned} H(t) = {} & \omega_p \left(b_k^\dagger b_k + \frac{3}{2} \right) + i\omega_p \sqrt{\frac{1}{2} V \omega_p} \left(b_k - b_k^\dagger \right) A_k(t) \\ & + \frac{1}{2} V \omega_p^2 \vec{A}(t)^2 + \omega_1 \left(c_k^\dagger c_k + \frac{3}{2} \right) + g \left(b_k c_k^\dagger + b_k^\dagger c_k \right) \\ & + ig \sqrt{\frac{1}{2} V \omega_p} \left(c_k - c_k^\dagger \right) A_k(t), \end{aligned} \quad (89)$$

where the original interaction term [11] is recognized.

The Heisenberg equations of motion follow as

$$\begin{aligned} \frac{db_k(t)}{dt} &= i[H, b_k] = -i\omega_p b_k - igc_k - \omega_p \sqrt{\frac{1}{2} V \omega_p} A_k(t), \\ \frac{dc_k(t)}{dt} &= i[H, c_k] = -i\omega_1 c_k - igb_k - g \sqrt{\frac{1}{2} V \omega_p} A_k(t). \end{aligned} \quad (90)$$

Note that different Cartesian components do not couple. In matrix form we have

$$\frac{d}{dt} \begin{pmatrix} b_k \\ c_k \end{pmatrix} = -i \begin{pmatrix} \omega_p & g \\ g & \omega_1 \end{pmatrix} \begin{pmatrix} b_k \\ c_k \end{pmatrix} - A_k(t) \sqrt{\frac{1}{2} V \omega_p} \begin{pmatrix} \omega_p \\ g \end{pmatrix}. \quad (91)$$

The evolution operator $\exp \mathcal{U}t$, where $\mathcal{U}$ denotes the appearing 2×2 matrix, which does not depend on the Cartesian index k, of the homogeneous differential equations can be explicitly calculated. First, one obtains for the frequency eigenvalues

$$\omega_\pm = \frac{1}{2} \left(\omega_p + \omega_1 \right) \pm \frac{1}{2} \sqrt{\left(\omega_p - \omega_1 \right)^2 + 4g^2}. \quad (92)$$

The corresponding eigenvectors read

$$\frac{1}{N} \begin{pmatrix} g \\ \omega_+ - \omega_p \end{pmatrix}, \qquad \frac{1}{N} \begin{pmatrix} \omega_- - \omega_1 \\ g \end{pmatrix}, \quad (93)$$

with normalization $N = \sqrt{(\omega_+ - \omega_p)^2 + g^2} = \sqrt{(\omega_- - \omega_1)^2 + g^2}$; note that one has $\omega_p - \omega_+ = \omega_- - \omega_1$. Herewith the evolution operator is constructed as

$$\exp \mathcal{U}t = \frac{1}{N^2} \begin{pmatrix} g^2 e^{-i\omega_+ t} + \tilde{\omega}^2 e^{-i\omega_- t} & g\hat{\omega} e^{-i\omega_+ t} + g\tilde{\omega} e^{-i\omega_- t} \\ g\hat{\omega} e^{-i\omega_+ t} + g\tilde{\omega} e^{-i\omega_- t} & \hat{\omega}^2 e^{-i\omega_+ t} + g^2 e^{-i\omega_- t} \end{pmatrix} \quad (94)$$

where $\tilde{\omega}$ denotes $\omega_- - \omega_1$ and $\hat{\omega}$ denotes $\omega_+ - \omega_p$. The solutions of the inhomogeneous differential equations are given by

$$\begin{pmatrix} b_k(t) \\ c_k(t) \end{pmatrix} = \exp \mathcal{U}t \begin{pmatrix} b_k(0) \\ c_k(0) \end{pmatrix} - \sqrt{\frac{1}{2} V \omega_p} \int_0^t dt' \exp[\mathcal{U}(t - t')] A_k(t') \begin{pmatrix} \omega_p \\ g \end{pmatrix}. \quad (95)$$

The expression for $\vec{\Pi}^0(t)$ trivially follows.

Finally, we specify the time-independent state as a product of eigenstates of the uncoupled Hamiltonians of the zero-modes and the additional degree of freedom respectively. Calculating the corresponding expectation value of the electric field operator yields after integrating by parts

$$\vec{E}_{\text{tot}}(t) = \vec{E}(t) + \frac{\omega_p}{N^2} \left(g^2 \vec{I}(\omega_+) + (\omega_- - \omega_1)^2 \vec{I}(\omega_-) \right), \quad (96)$$

where we have defined

$$\vec{I}(\xi) = \int_0^t dt' \sin \xi (t' - t) \vec{E}(t'). \quad (97)$$

In comparison with the plasma response without additional interaction, there appear two relevant frequencies which are determined by the original plasma frequency, the oscillator frequency and the coupling constant. The response at such a frequency is similar to the original one. Thus two resonances appear. This is explicitly seen by considering a harmonic external field $\vec{E}(t) = \vec{E}_0 \cos\omega t$. Performing the integrals analytically gives for this case

$$\vec{E}_{\text{tot}}(t) = \vec{E}_0\Bigg[\frac{g^2}{N^2}\frac{\omega_+\omega_p}{\omega_+^2-\omega^2}\cos\omega_+ t + \frac{(\omega_- - \omega_1)^2}{N^2}\frac{\omega_p\omega_-}{\omega_-^2-\omega^2}\cos\omega_- t + \left(1-\frac{g^2}{N^2}\frac{\omega_+\omega_p}{\omega_+^2-\omega^2}-\frac{(\omega_- - \omega_1)^2}{N^2}\frac{\omega_-\omega_p}{\omega_-^2-\omega^2}\right)\times\cos\omega t\Bigg]. \tag{98}$$

8. Plasma, Bath and External Field

Finally, we extend our plasma model in order to describe the expected damping and fluctuations in realistic systems. To this end, we explicitly include the surrounding medium, or *bath*, as part of the system. In other words, additional interactions are taken into account; in principle we follow the approach of van Kampen [11]. In order not to overload the notation we consider only one Cartesian component of the complete model. It is assumed that the bath degrees of freedom as well as the external electric field couple to this component. The two other components then trivially decouple. Therefore the Cartesian index is omitted from now on. The essential physics evidently is captured in this way. Alternatively, one may consider the total Hamiltonian of [1] completely describing the plasma in interaction with the dynamical electromagnetic fields. This is, however, beyond the scope of our study.

8.1. Plasma and Bath. The Hamilton operator consists of three parts:

$$H = \frac{1}{2}V\omega_0^2\left(A^0 - \frac{1}{Ne}P\right)^2 + \frac{1}{2V}(\Pi^0)^2 + H_B + H_I, \tag{99}$$

where the first terms describe again the zero-modes of the internal electromagnetic field coupled to the CM momentum of the ions. Here we denote the plasma frequency by ω_0. The free bath Hamiltonian is taken as the sum of a number of harmonic oscillators with frequencies ω_n, that is,

$$H_B = \frac{1}{2}\sum_n(p_n^2 + \omega_n q_n^2) = \sum_n \omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right) \tag{100}$$

with coordinates q_n and momenta p_n, $[p_n, q_m] = -i\delta_{nm}$. The annihilation and creation operators are defined as

$$a_n = \frac{1}{\sqrt{2\omega_n}}(p_n - i\omega_n q_n), \quad a_n^\dagger = \frac{1}{\sqrt{2\omega_n}}(p_n + i\omega_n q_n). \tag{101}$$

It follows that $[a_n, a_m^\dagger] = \delta_{nm}$. We also define such operators for the zero-mode:

$$a_0 = \frac{1}{\sqrt{2V\omega_0}}(\Pi^0 - iV\omega_0 A^0), \quad a_0^\dagger = \frac{1}{\sqrt{2V\omega_0}}(\Pi^0 + iV\omega_0 A^0). \tag{102}$$

Van Kampen defines the interaction as [11]

$$H_I^0 = \sum_n g_n\left(a_n a_0^\dagger + a_n^\dagger a_0\right) = \frac{1}{\sqrt{V\omega_0}}\sum_n\frac{g_n}{\sqrt{\omega_n}}(p_n\Pi^0 + V\omega_0\omega_n q_n A^0) \tag{103}$$

with coupling strengths g_n. This interaction Hamiltonian is not invariant under displacements. Just as in the previous section, we therefore insert the symmetric "minimal substitution" term and use the interaction

$$H_I = \frac{1}{\sqrt{V\omega_0}}\sum_n\frac{g_n}{\sqrt{\omega_n}}\left(p_n\Pi^0 + V\omega_0\omega_n q_n\left(A^0 - \frac{1}{Ne}P\right)\right) = \sum_n g_n\left(a_n a_0^\dagger + a_n^\dagger a_0\right) - \frac{i}{Ne}\sqrt{\frac{1}{2}V\omega_0}\sum_n g_n\left(a_n - a_n^\dagger\right)P. \tag{104}$$

Thus the bath interacts with the CM degree of freedom as well. The Hamiltonian can now be now written as

$$H = \omega_0\left(a_0^\dagger a_0 + \frac{1}{2}\right) - \frac{i\omega_0}{Ne}\sqrt{\frac{1}{2}V\omega_0}\left(a_0 - a_0^\dagger\right)P + \sum_n\omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right) + \sum_n g_n\left(a_n a_0^\dagger + a_n^\dagger a_0\right) - \frac{i}{Ne}\sqrt{\frac{1}{2}V\omega_0}\sum_n g_n\left(a_n - a_n^\dagger\right)P + \frac{1}{2Nm}P^2 \tag{105}$$

and can be analyzed analogous to the procedure of [11].

8.2. Inclusion of an External Electric Field. We proceed by including a homogeneous external field (12); the Hamiltonian becomes explicitly time-dependent:

$$\begin{aligned}H(t) = {}&\omega_0\left(a_0^\dagger a_0 + \frac{1}{2}\right) - \frac{i\omega_0}{Ne}\sqrt{\frac{1}{2}V\omega_0}\left(a_0 - a_0^\dagger\right)P \\ &+ i\omega_0\sqrt{\frac{1}{2}V\omega_0}\left(a_0 - a_0^\dagger\right)A(t) \\ &- \frac{V\omega_0^2}{Ne}PA(t) + \frac{1}{2}V\omega_0^2A^2(t) + \sum_n\omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right) \\ &+ \sum_n g_n\left(a_n a_0^\dagger + a_n^\dagger a_0\right) \\ &- \frac{i}{Ne}\sqrt{\frac{1}{2}V\omega_0}\sum_n g_n\left(a_n - a_n^\dagger\right)P + \frac{1}{2Nm}P^2 \\ &+ i\sqrt{\frac{1}{2}V\omega_0}\sum_n g_n\left(a_n - a_n^\dagger\right)A(t).\end{aligned} \tag{106}$$

Since the CM coordinate is cyclic, $[H, P] = 0$, the conjugate momentum P is a constant of motion $P \rightarrow k_j = 2nj/L$. After inserting this in the Hamiltonian, we define the "shifted" operators b_0 and $b_0^\dagger$:

$$\begin{aligned} a_0 &= b_0 - i\alpha, \\ a_0^\dagger &= b_0^\dagger + i\alpha, \end{aligned} \tag{107}$$

with $\alpha = \alpha(j) = (k_j/Ne)\sqrt{(1/2)V\omega_0}$. Some algebra leads to

$$\begin{aligned} H(t) = {} & \omega_0\left(b_0^\dagger b_0 + \frac{1}{2}\right) + i\omega_0\sqrt{\frac{1}{2}V\omega_0}\left(b_0 - b_0^\dagger\right)A(t) \\ & + \frac{1}{2}V\omega_0^2A^2(t) + \sum_n \omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right) \\ & + \sum_n g_n\left(a_n b_0^\dagger + a_n^\dagger b_0\right) \\ & + i\sqrt{\frac{1}{2}V\omega_0}\sum_n g_n\left(a_n - a_n^\dagger\right)A(t). \end{aligned} \tag{108}$$

The equations of motion for the Heisenberg operators follow as

$$\begin{aligned} \frac{db_0(t)}{dt} &= i[H, b_0] = -i\omega_0 b_0 - i\sum_m g_m a_m - \omega_0\sqrt{\frac{1}{2}V\omega_0}A(t), \\ \frac{da_n(t)}{dt} &= i[H, a_n] = -i\omega_n a_n - ig_n b_0 - g_n\sqrt{\frac{1}{2}V\omega_0}A(t). \end{aligned} \tag{109}$$

The homogeneous differential equations are implicitly solved in [11]:

$$\begin{aligned} b_0(t) &= U(t)b_0(0) + \sum_m V_m(t)a_m(0), \\ a_n(t) &= W_n(t)b_0(0) + \sum_m S_{nm}(t)a_m(0). \end{aligned} \tag{110}$$

The corresponding evolution operator is given by

$$\exp \mathcal{M}t = \begin{pmatrix} U(t) & \vec{V}^{\mathrm{tr}}(t) \\ \vec{W}(t) & S(t) \end{pmatrix}. \tag{111}$$

The matrix $\mathcal{M}$ is implicitly defined in (109). For $b_0(t)$ we consequently get as solution of the inhomogeneous differential equation:

$$b_0(t) = U(t)b_0(0) + \sum_m V_m(t)a_m(0) - \sqrt{\frac{1}{2}V\omega_0}\eta(t), \tag{112}$$

where we have defined

$$\eta(t) = \omega_0\int_0^t U(t-t')A(t')dt' + \sum_m g_m\int_0^t V_m(t-t')A(t')dt'. \tag{113}$$

The zero-mode electric field operator follows as

$$\Pi^0 = \sqrt{\frac{1}{2}V\omega_0}\left(b_0 + b_0^\dagger\right). \tag{114}$$

We take as time-independent state $\Psi = \Psi_0 \cdot \psi_B$, with Ψ_0 being an eigenstate of the plasma Hamiltonian and ψ_B being an eigenstate of the bath Hamiltonian, that is, the product of the shifted zero-mode oscillator, the plane wave and bath harmonic oscillator wave functions. Then we get for the expectation value of the total electric field:

$$E_{\mathrm{tot}} = -\frac{1}{V}\langle\Pi^0\rangle + E(t) = \frac{1}{2}\omega_0\left(\eta(t) + \eta^*(t)\right) + E(t). \tag{115}$$

8.3. Evolution Operator. Let us address the solution of the equations of motion (cf. (109)) in some more detail and explicitly construct the evolution operator. In principle we follow the approach of van Kampen [11].

8.3.1. Exact Developments. The set of (109) is a solved with the *ansatz* $e^{-i\lambda t}$ for the time dependence of all modes. This leads to the eigenvalue equation:

$$\lambda - \omega_0 = \sum_n \frac{g_n^2}{\lambda - \omega_n} \tag{116}$$

with a sequence of solutions λ_ν. Given the number of bath oscillators, their frequencies, and the coupling constants, they can be determined numerically. The corresponding eigenvectors are denoted by $\vec{X}_\nu = (X_{0\nu}, X_{n\nu})$ and fulfill

$$X_{n\nu} = \frac{g_n}{\lambda_\nu - \omega_n}X_{0\nu}. \tag{117}$$

The first component $X_{0\nu}$ follows from the imposed normalization condition; it explicitly reads

$$1 = X_{0\nu}^2 + \sum_n X_{n\nu}^2 = X_{0\nu}^2\left\{1 + \sum_n \frac{g_n^2}{(\lambda_\nu - \omega_n)^2}\right\}. \tag{118}$$

The eigenvectors become orthonormal:

$$\vec{X}_\nu \cdot \vec{X}_\mu = X_{0\nu}X_{0\mu} + \sum_n X_{n\nu}X_{n\mu} = \delta_{\mu\nu}, \tag{119}$$

and the closure relations

$$\sum_\nu X_{0\nu}^2 = 1, \qquad \sum_\nu X_{n\nu}X_{m\nu} = \delta_{mn}, \qquad \sum_\nu X_{0\nu}X_{n\nu} = 0 \tag{120}$$

can be shown to hold true.

The operators b_0 and a_n are linear combinations of the normal modes:

$$b_0(t) = \sum_\nu c_\nu X_{0\nu}e^{-i\lambda_\nu t}, \qquad a_n(t) = \sum_\nu c_\nu X_{n\nu}e^{-i\lambda_\nu t} \tag{121}$$

with superposition constants c_ν. Using orthonormality (119), they follow from the initial values as

$$c_\nu = X_{0\nu}b_0(0) + \sum_n X_{n\nu}a_n(0). \tag{122}$$

This indeed yields the solution (110) with the explicit expressions:

$$U(t) = \sum_\nu X_{0\nu}^2 e^{-i\lambda_\nu t}, \qquad V_n(t) = \sum_\nu X_{0\nu} X_{n\nu} e^{-i\lambda_\nu t},$$
$$W_m(t) = V_m(t), \qquad S_{mn}(t) = \sum_\nu X_{n\nu} X_{m\nu} e^{-i\lambda_\nu t}. \tag{123}$$

Thus, we have constructed the evolution operator (111). Note that closure (120) guarantuees the necessary property $\exp[\mathcal{M}(t = 0)] = \mathcal{I}$. These exact results can be used in order to calculate the expectation value of the electric field as outlined above. Note that the bath degrees of freedom are eventually eliminated by our choice of the state. Since a realistic number of bath oscillators is huge, the numerical computations are expected to be time/memory consuming and therefore challenging. We have not yet attempted to perform this task.

8.3.2. Approximate Evolution Operator. In [11] further developments on the operator level preclude the elimination of the bath degrees of freedom in the way we have done. In order to proceed, an approximate, simpler form of the evolution operator is derived. The appearing summation over many bath modes is effectively done. This approximation may be also be interesting for our purposes because the numerical work mentioned above is considerably reduced. We therefore present a slightly modified derivation of the results in [11]; here we do not explicitly need the concept of a "strength function." It starts by defining the function

$$G(z) = z - \omega_0 - \sum_n \frac{g_n^2}{z - \omega_n}. \tag{124}$$

Its zeros are the eigenfrequencies λ_ν of the total system (cf. (116)). At such a zero one gets for the derivative

$$G'(\lambda_\nu) = 1 + \sum_n \frac{g_n^2}{(\lambda_\nu - \omega_n)^2} = \frac{1}{X_{0\nu}^2}, \tag{125}$$

where the normalization condition (118) has been used. Using complex function theory yields for $t > 0$

$$U(t) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-ixt+\epsilon t}}{G(x + i\epsilon)} dx, \tag{126}$$

where, as usual, ϵ is small and positive. This is still exact. At this point additional assumptions are made. First, ϵ is small compared to the scale over which the coupling constants vary. This can be formulated in terms of the abovementioned strength function [11]. Secondly, ϵ is *large* compared to the distance between the bath frequencies ω_n. Finally, and "more seriously," it is assumed that the interaction is weak, meaning small couplings g_n (and, concomitantly, small strength function). This implies that $G(x + i\epsilon)$ can be approximated by replacing x by ω_0 in the summation term of (124). The result can be written as

$$G(x + i\epsilon) \approx x - \omega_0' + i\Gamma \tag{127}$$

with shifted frequency ω_0'

$$\omega_0' = \omega_0 + \sum_n \frac{g_n^2(\omega_0 - \omega_n)}{(\omega_0 - \omega_n)^2 + \epsilon^2}, \tag{128}$$

and damping Γ

$$\Gamma = \epsilon\left(1 + \sum_n \frac{g_n^2}{(\omega_0 - \omega_n)^2 + \epsilon^2}\right). \tag{129}$$

Hence one obtains

$$U(t) \approx e^{-i\omega_0' t - \Gamma t}, \tag{130}$$

and the same approximations yield

$$V_n(t) = W_n(t) \approx \frac{g_n}{\omega_0' - \omega_n - i\Gamma}\left\{e^{-i\omega_0' t - \Gamma t} - e^{-i\omega_n t}\right\}. \tag{131}$$

8.4. Finite Temperature Considerations

8.4.1. Initial State—Occupation Number. Instead of selecting the initial state as product of the respective eigenstates of the plasma and bath Hamiltonians, one may assume the bath at $t = 0$ to be in thermal equilibrium with the temperature T [11]. The plasma initial state is supposed to be unaltered. The corresponding expectation values of an operator O are generally found as

$$\langle O \rangle = \frac{\mathrm{Tr}\left(Oe^{-\beta H}\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)} \tag{132}$$

with inverse temperature $\beta^{-1} = kT$. Recall that the trace of an operator can be computed using any complete orthonormal set of states $|\alpha\rangle$:

$$\mathrm{Tr}(O) = \sum_\alpha \langle \alpha | O | \alpha \rangle. \tag{133}$$

The denominator of (132) is the well-known partition function of the canonical ensemble. For the chosen bath, a collection of harmonic oscillators, it can readily be calculated using the harmonic oscillators states:

$$\mathrm{Tr}\left(e^{-\beta H_B}\right) = \prod_n \frac{e^{-(1/2)\beta\omega_n}}{1 - e^{-\beta\omega_n}}. \tag{134}$$

The average occupation number of the bath degrees of freedom follows as

$$\left\langle a_n^\dagger(0) a_m(0) \right\rangle = \delta_{mn} \frac{1}{e^{\beta\omega_n} - 1}. \tag{135}$$

With this result and (110) van Kampen computes the expectation value of $a_0^\dagger(t) a_0(t)$ (corresponding to the degree of freedom in the system, in our case the zero-mode with operators $b_0^\dagger(t), b_0(t)$) and obtains a nontrivial temperature dependence [11].

8.4.2. Electric Field. We are concerned with the expectation value of the electric field operator, which is proportional to $a_0(t) + a_0^\dagger(t)$. In principle, the field may get temperature dependent contributions due to the bath—as in the previous example. These possibly contributing terms, however, are linear in the operators $a_n^\dagger(0)$ and $a_n(0)$ (cf. (107) and (110)). The thermal equilibrium expectation values of the bath annihilation operators vanish:

$$\langle a_n \rangle = \frac{\mathrm{Tr}\left(a_n e^{-\beta H_B}\right)}{\mathrm{Tr}\left(e^{-\beta H_B}\right)} = 0. \tag{136}$$

This can be understood by once more inserting the eigenstates of the bath Hamiltonian H_B and formally expanding it. Each individual term in the trace contains an odd number of annihilation operators because of the additional operator a_n. Consequently, each expectation value in all eigenstates is zero and this yields a vanishing trace. Analogously, we find that $\langle a_n^\dagger \rangle = 0$. Therefore, no temperature dependence in the expectation value of the electric field is present in this model. The results are equal to those obtained with an eigenstate of the bath Hamiltonian as initial state. This result is almost trivial for zero external electric field. It is also valid, however, for nonzero, spatially constant, time-dependent external electric fields. In particular, the induced electric field—stemming from the inhomogeneous terms in the equations of motion and in principle determined by (113)—is not modified.

8.4.3. Energy. Finally, we derive the expression for the expectation value of the Hamilton operator (109), interpretable as the time-dependent energy $\mathcal{E}(t)$. The initial state is chosen as above, meaning that the bath is in thermal equilibrium at $t = 0$. The harmonic oscillator state is labelled here with quantum number n_0. In addition to the time-development of b_0 given by (112) and (113), we also need

$$a_n(t) = W_n(t) b_0(0) + \sum_m S_{nm}(t) a_m(0) - \sqrt{\frac{1}{2} V \omega_0} \zeta_n(t) \tag{137}$$

with the definition

$$\begin{aligned}\zeta_n(t) = {} & \omega_0 \int_0^t W_n(t-t') A(t') dt' \\ & + \sum_m g_m \int_0^t S_{nm}(t-t') A(t') dt'.\end{aligned} \tag{138}$$

For the occupation number one then gets

$$\begin{aligned}\left\langle b_0^\dagger(t) b_0(t) \right\rangle = {} & n_0 U^*(t) U(t) + \sum_m V_m^*(t) V_m(t) \left(e^{\beta\omega_m} - 1\right)^{-1} \\ & + \frac{1}{2} V \omega_0 \eta^*(t) \eta(t).\end{aligned} \tag{139}$$

If we omit the contribution due to the external field, that is taking $\eta = 0$, we indeed reproduce the abovementioned result of [11]. The expectation value of the complete Hamiltonian $\langle H(t) \rangle$ follows as

$$\begin{aligned}\mathcal{E}(t) = {} & \omega_0 \Big\{ n_0 U^*(t) U(t) + \sum_m V_m^*(t) V_m(t) \left(e^{\beta\omega_m} - 1\right)^{-1} \\ & \quad + \frac{1}{2} V \omega_0 \eta^*(t) \eta(t) + \frac{1}{2} \Big\} \\ & + \frac{1}{2} V \omega_0^2 \Big\{ i(\eta^*(t) - \eta(t)) A(t) + A^2(t) \Big\} \\ & + \frac{1}{2} V \omega_0 A(t) \sum_n g_n i(\zeta_n^*(t) - \zeta_n(t)) \\ & + \sum_n \omega_n \Big\{ n_0 W_n^*(t) W_n(t) + \frac{1}{2} V \omega_0 \zeta_n^*(t) \zeta_n(t) + \frac{1}{2} \Big\} \\ & + \sum_n g_n \Big\{ n_0 (W_n^*(t) U(t) + W_n(t) U^*(t)) \\ & \quad + \frac{1}{2} V \omega_0 (\zeta_n^*(t) \eta(t) + \zeta_n(t) \eta^*(t)) \Big\} \\ & + \sum_{nm} \Big\{ g_n (S_{nm}^*(t) V_m(t) + S_{nm}(t) V_m^*(t)) \\ & \quad + \omega_n S_{nm}^*(t) S_{nm}(t) \Big\} \left(e^{\beta\omega_m} - 1\right)^{-1}.\end{aligned} \tag{140}$$

This result is exact and can be implemented in an algorithm. Alternatively, one may use the approximations given above (cf. (130) and (131)) supplemented with the analogous expression for S. The corresponding approximation for the occupation number is explicitly given in [11].

9. Epilogue

The possible effects of electromagnetic fields on biological systems, for example, human beings, are a topical subject. Although possibly relevant physics has already been addressed some time ago [2], there are hardly biophysical mechanisms known and accepted to date. This paper may describe a coupling phenomenon with consequences, in the sense that the internal electric field can be much larger than the external one. It is based on the fact that there are "free" ions in intracellular and extracellular fluid. Consequently, one may describe these ions as a low-density plasma.

The response of an ion plasma to external electromagnetic fields has therefore been investigated in this study. To this end, the theory developed in [1] for electron plasmas has been applied. It focusses on the zero-momentum photons, center of mass motion and spatially constant electric fields in periodic structures. Typical plasma oscillations show up. Since the length scale of our problem is much smaller than the wavelengths of typical external fields, this approach is appropriate. The basic quantum mechanical resonance mechanism of [1], demonstrated for harmonic fields, remains essentially the same. We have shown that resonances appear for several selected waveforms, as well as for general bandpass signals. The appearing resonance and plasma frequencies, however, are quite different because the pertinent physical parameters, density and mass, of the ions differ from those of electrons. We have explicitly estimated these frequencies for the relevant ions and found them to be in the region 0.5 GHz–400 GHz.

The chosen formulation of the plasma is translationally invariant [1] and, concomitantly, a homogeneously distributed background charge is assumed. Oppositely charged ions in the intra- and extracelluar fluids are supposed to be related to this background. In our study, we have extended this framework to a two-species plasma because it appears to be more appropriate for our biophysical application. Such a plasma actually may or may not be electrically neutral; in the first case a background charge distribution is absent. The effects of coupling an external electric field to the two-species plasma are found to be essentially the same as in the simple plasma.

Further investigations of the effects of electromagnetic fields are done by switching to the Heisenberg picture of quantum mechanics, which has turned out to be more convenient for our purposes. Earlier results are readily reproduced. The time-dependent energy of the plasma is also calculated and expressed in induced field and current. The plasma is extended with an additional degree of freedom, interacting with the zero-mode photon. The relic of the original gauge symmetry, that is the displacement symmetry [1], dictates an additional interaction involving the CM motion of the ions. The calculations yield two resonance frequencies instead of one.

The chosen plasma model is certainly not completely realistic in the sense that damping and fluctuations, expected in real systems, are not present. This is also evident from the underlying complete theory [1]: we have restricted ourselves to the zero-mode Hamiltonian and ignored the other Hamiltonians describing relative motion, finite momentum photons and additional couplings. It is beyond the purpose of this research to study the effects of these extra degrees of freedoms; such investigations are actually initiated in [1]. Instead we choose the approach of [11] to include a surrounding medium or bath. An exact expression for the internal electric field is derived in terms of an evolution operator. In order to generate numerical results, however, further algorithmic implementation is necessary. This is planned for the nearby future. In the present study, an approximate weak coupling form [11] of the response is constructed. The resonance frequency is shifted and damping indeed appears. Finally, it is shown that assuming thermal equilibrium, the expectation value of the electric field is not changed. The energy of the complete system is found to depend on the temperature.

The conclusion of this work is that in the chosen plasma model and its extensions resonance effects appear, which induce currents and generate large internal electric fields. The energy of the plasma is increased as well. Quantum mechanical plasma oscillations are responsible for this physical amplifying mechanism. Note that the response is linear and no demodulation of the external field takes place. On the other hand, a resonant harmonic field becomes amplitude modulated within the plasma. Whether such a large electric field has consequences in biological systems is, of course, an open question. It is furthermore not clear that the plasma model indeed grasps the essential physics of ions in intra- and extracellular fluids responding to external electromagnetic fields. Nevertheless we believe that we have found an interesting possibility of unexpected electromagnetic effects in biophysics.

Acknowledgments

The author thanks M. Thies and S. H. J. A. Vossen for useful discussions and critical readings of the manuscript. This work has been supported by the Dutch Ministry of Defence, Grant W04/06 "RF Biological Effects."

References

[1] F. Lenz, H. W. L. Naus, K. Ohta, and M. Thies, "Zero modes and displacement symmetry in electrodynamics," *Annals of Physics*, vol. 233, no. 1, pp. 51–81, 1994.

[2] H. Fröhlich, "The biological effects of microwaves and related questions," *Advances in Electronics and Electron Physics*, vol. 53, pp. 85–152, 1980.

[3] F. Apollonio, M. Liberti, G. d'Lnzeo, and L. Tarricone, "Integrated models for the analysis of biological effects of EM fields used for mobile communications," *IEEE Transactions on Microwave Theory and Techniques*, vol. 48, no. 11, pp. 2082–2093, 2000.

[4] K. R. Foster and M. H. Repacholi, "Biological effects of radiofrequency fields: the electroencephalogram during a visual working memory task," *Rad Research*, vol. 162, p. 219, 2004.

[5] A. C. Green, I. R. Scott, R. J. Gwyther, et al., "An investigation of the effects of TETRA RF fields on intracellular calcium in neurones and cardiac myocytes," *International Journal of Radiation Biology*, vol. 81, no. 12, pp. 869–885, 2005.

[6] L. J. Challis, "Mechanisms for interaction between RF fields and biological tissue," *Bioelectromagnetics*, vol. 26, supplement 7, pp. S98–S106, 2005.

[7] M. Simeonova and J. Gimsa, "The influence of the molecular structure of lipid membranes," *Bioelectromagnetics*, vol. 27, no. 8, pp. 652–666, 2006.

[8] J. R. Jauchem, "Effects of low-level radio-frequency energy on human cardiovascular," *International Journal of Hygiene and Environmental Health*, vol. 211, pp. 1–29, 2008.

[9] N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, Thomson Learning, Boston, Mass, USA, 1976.

[10] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, Academic Press, San Diego, Calif, USA, 4th edition, 1995.

[11] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, Elsevier, Amsterdam, The Netherlands, 3rd edition, 1997.