Generalized MTSFM Signals Expressed as Complex Fourier Series using Generalized Bessel Functions of Anger Type: Representation and Application

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***Abstract*—The classical multi-tone sinusoidal frequency modulation (MTSFM) is a Fourier expansion of the instantaneous frequency or modulation function of a signal, where the Fourier-expansion period equals the pulse width. We extended or generalized the MTSFM by introducing Fourier expansions with arbitrary periods and by expanding the phase instead of the instantaneous frequency. This generalization allows the representation of less smooth signals and avoids significant root mean square (RMS) bandwidth (and swept bandwidth) increase. In our (numerical) analysis we computed the signal and its corresponding Fourier transform (FT) and auto-correlation function (ACF) by directly evaluating the MTSFM expansion and calculating the integrals in the FT and ACF by Riemann sums. Instead, an alternative approach in the classical MTSFM has been developed, in which the signal is represented by a complex Fourier expansion with generalized Bessel functions (GBFs) of Anger type as coefficients, briefly called a Jacobi-Anger expansion. In this paper we derive expressions for the Jacobi-Anger expansions of a generalized MTSFM signal and its FT and ACF, both without and with window, and present approaches for analytical and numerical evaluation of these expansions. We apply our theory to a classical example MTSFM signal of which we present the numerically evaluated signal, FT, and ACF by Jacobi-Anger expansions, and we comment on the complexity of the expansions in comparison to direct evaluation of the generalized MTSFM signal, its FT, and its ACF, as well as on the additional insight the expansions provide in the behaviour of these quantities. Finally, we describe our future analysis with application of the Jacobi-Anger expansions in optimization.**

***Index Terms*—Signal analysis, continuous phase modulation, OFDM, Fourier series, chirp modulation**

I. Introduction

In the past twelve years several theory and application papers have appeared on the MTSFM, which is a Fourier expansion of the instantaneous frequency or modulation function of a signal. For an overview we refer to [1], which presents also history of the MTSFM and embeds it in broader classes of waveforms, including the constant-envelope orthogonal frequency division multiplexing (CE-OFDM) class. In [2] we generalized the classical MTSFM signal model by introducing Fourier expansions with arbitrary periods instead of only the period equal to the pulse width as in the classical MTSFM, see e.g. [1], [3]–[9]. Also, we expanded the instantaneous phase instead of the instantaneous frequency, which allows for the representation of less smooth signals, fits better the nature of current arbitrary waveform generators (AWGs), where discrete samples of the phase of a signal are prescribed, and avoids significant RMS bandwidth and swept bandwidth increase. In our (numerical) analysis we computed the signal and its FT and ACF by direct integration of the generalized-MTSFM represented integrands. Instead, in [1], [3]–[9] the MTSFM signal and its FT and ACF are written as complex (finite) Fourier expansions with generalized Bessel functions (GBFs) of Anger type as expansion coefficients, also shortly called Jacobi-Anger expansions [12, p. 1164]. This description would greatly simplify analysis of the MTSFM model [1, p. 1278].

In this paper we derive similar expressions for our generalized MTSFM signal model and detail their evaluation. To this end we introduce first the required GBF expressions. Next, we employ these expressions to obtain Jacobi-Anger expansions of a generalized MTSFM signal and its FT and ACF, both without and with window. The integrals in the FT and ACF expressions without window can be evaluated analytically, such that only a single and a double (infinite) sum remains, respectively, as in the classical MTSFM [1]. For the expressions with window, we describe how they can be computed in an efficient manner. Note that, to the best of our knowledge, the classical MTSFM papers with analysis of Jacobi-Anger expansions [1], [3]–[9] and two GBF papers [10], [11] do not provide details regarding the computation of the GBFs and the Jacobi-Anger expansions despite results in e.g. [1] including a Tukey window. Having established the theory, we present numerical results for a representation example in [2, Sec. III.A]. We end with conclusions and a summary of our future work to employ the Jacobi-Anger expansions in other representation examples and in optimization.

II. Prerequisites

Mixed-type GBFs (MT-GBFs, with cosine and sine terms) can for example be introduced by defining

$$\exp\left(\sum_{m=1}^{M}\left[a_m \cos m\theta + b_m \sin m\theta\right]\right) = \sum_{n=-\infty}^{\infty} J_n(-j\boldsymbol{b};\boldsymbol{a})\, e^{jn\theta} \quad (1)$$

analogous to [13, Eq. (5.6)]. Then,

$$J_n(\boldsymbol{b};\boldsymbol{a}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(\sum_{m=1}^{M}\left[a_m \cos m\theta + jb_m \sin m\theta\right] - jn\theta\right) d\theta \quad (2)$$

analogous to [14, Eq. (4.9)] with $\alpha_m \to b_n$ and $\beta_m \to a_n$, which is called a (MT-)GBF of Anger type. Thus,

$$J_n(\boldsymbol{b};j\boldsymbol{a}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(j\left\{\sum_{m=1}^{M}\left[a_m \cos m\theta + b_m \sin m\theta\right] - n\theta\right\}\right) d\theta \quad (3)$$

which are according to (1) the coefficients of the complex Fourier series of

$$\exp\left(j\sum_{m=1}^{M}\left[a_m \cos m\theta + b_m \sin m\theta\right]\right) = \sum_{n=-\infty}^{\infty} J_n(\boldsymbol{b};j\boldsymbol{a}) e^{jn\theta} \quad (4)$$

The even and odd parts of the instantaneous phase in (4) are obtained by setting $\boldsymbol{b} = \boldsymbol{0}$ and $\boldsymbol{a} = \boldsymbol{0}$, respectively. Then,

$$J_n(\boldsymbol{0};j\boldsymbol{a}) = I_n(j\boldsymbol{a}), \qquad J_n(\boldsymbol{b};\boldsymbol{0}) = J_n(\boldsymbol{b}) \quad (5)$$

where J_n and I_n are the GBF and modified GBF, respectively, which are the multi-variable analogues of the classical (single-variable) Bessel function and modified Bessel function. In the following we will refer to the MT-GBF as GBF.

III. Application to the Generalized or Extended MTSFM Signal Model

In the generalized MTSFM a pulse is modeled by

$$g(t) = \exp(j\varphi(t))\, 1_{\Omega_t}(t) = e^{ja_0} 1_{\Omega_t}(t) \exp\left(j\sum_{q=1}^{Q}\left[a_q \cos\left(2\pi q\frac{t}{T_\mathrm{p}}\right) + b_q \sin\left(2\pi q\frac{t}{T_\mathrm{p}}\right)\right]\right) \quad (6)$$

where 1_{Ω_t} is the characteristic function, which is one on the interval Ω_t and zero elsewhere. Without compromise, we set $1_{\Omega_t} = [0, T]$ as in [2]. According to (4)

$$g(t) = e^{ja_0} 1_{[0,T]}(t) \sum_{n=-\infty}^{\infty} J_n(\boldsymbol{b};j\boldsymbol{a}) \exp\left(jn\frac{2\pi t}{T_\mathrm{p}}\right) \quad (7)$$

where $J_n(\boldsymbol{b};j\boldsymbol{a})$ is given by the integral expression (3) with M replaced by Q. In [1, paragraph below Eq. (22)] invariance of the expansion coefficients of the classical MTSFM is demonstrated for fixed time-bandwidth product (TBP). For the generalized MTSFM, consider two pulse modulations with the same TBP, but with different durations T_1 and $T_2 = T_1/\xi$. Since TBP $= T_n \Delta f_{\mathrm{swept},n}$, we obtain $\Delta f_{\mathrm{swept},2} = \xi \Delta f_{\mathrm{swept},1}$. Also, from the definitions of the swept bandwidth and the modulation function in [2, Eqs. (3), (12)], it follows that

$$a_{q,2} = \xi a_{q,1}\frac{T_{\mathrm{p},2}}{T_{\mathrm{p},1}}, \qquad b_{q,2} = \xi b_{q,1}\frac{T_{\mathrm{p},2}}{T_{\mathrm{p},1}} \quad (8)$$

with $T_{\mathrm{p},n}$ the periods of the two generalized MTSFM expansions. Hence, $a_{q,2} = a_{q,1}$ and $b_{q,2} = b_{q,1}$ if $T_{\mathrm{p},2} = T_{\mathrm{p},1}/\xi$.

The spectrum of g is given by $\mathcal{F}\{g(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)\, \exp(-j\omega t)\, dt$. Substituting (7) and $\omega = 2\pi f$ in this FT and reversing sum and integral, we obtain

$$\mathcal{F}\{g(t)\} = \frac{e^{ja_0}T}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} J_n(\boldsymbol{b};j\boldsymbol{a})\, \mathrm{sinc}\left(\pi T\left(\frac{n}{T_\mathrm{p}} - f\right)\right) \exp\left(j\pi T\left(\frac{n}{T_\mathrm{p}} - f\right)\right) \quad (9)$$

The autocorrelation function (ACF) is given by

$$R(\tau) = \int_{\max(0,\tau)}^{\min(T,T+\tau)} g(t) g^*(t-\tau)\, dt \quad (10)$$

Substituting (7) in (10) and reversing sums and integral, we obtain

$$R(\tau) = (T - |\tau|) \sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} J_n(\boldsymbol{b};j\boldsymbol{a}) J_{n'}^*(\boldsymbol{b};j\boldsymbol{a}) \exp\left(\frac{j\pi(n+n')\tau}{T_\mathrm{p}}\right) \exp\left(\frac{j\pi(n-n')T}{T_\mathrm{p}}\right) \mathrm{sinc}\left(\frac{\pi(n-n')(T-|\tau|)}{T_\mathrm{p}}\right) \quad (11)$$

Introducing a continuous window function $w(t)$ in the FT or spectrum, we obtain

$$\mathcal{F}\{g(t)\} = \frac{e^{ja_0}}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} J_n(\boldsymbol{b};j\boldsymbol{a}) \int_0^T w(t) \exp\left(2\pi j\left(\frac{n}{T_\mathrm{p}} - f\right)t\right) dt \quad (12)$$

For specific windows the integral can be calculated analytically. For example, for a Tukey window [2, Eqs. (14)], we need to split the integral in three different portions. Though this calculation is not complex, we will compute the integral numerically as explained in the next section.

Analogously, introducing the window w in the ACF in (10), substituting the Jacobi-Anger expansion (7) of the signal g also in (10), and reversing sums and integral, we obtain

$$R(\tau) = \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} J_n(\boldsymbol{b}; j\boldsymbol{a}) J_{n'}^*(\boldsymbol{b}; j\boldsymbol{a})\, e^{\frac{2\pi j n' \tau}{T_\mathrm{p}}}$$
$$\int_0^T 1_{[lb(\tau), ub(\tau;T)]}(t)\, w(t)\, w^*(t-\tau)\, e^{\frac{2\pi j(n-n')t}{T_\mathrm{p}}} dt,$$
$$-T \leq \tau \leq T \quad (13)$$

where $lb(\tau) = \max(0, \tau)$ and $ub(\tau; T) = \min(T, T + \tau)$. For rapid computation it is convenient to introduce $m = n - n'$ and replacing n' by $m - n$. Then,

$$R(\tau) = \sum_{m=-\infty}^{\infty} I(\tau; m, T, T_\mathrm{p})$$
$$\sum_{n=-\infty}^{\infty} J_n(\boldsymbol{b}; j\boldsymbol{a}) J_{n-m}^*(\boldsymbol{b}; j\boldsymbol{a}) e^{\frac{2\pi j(n-m)\tau}{T_\mathrm{p}}} \quad (14)$$

where

$$I(\tau; m, T, T_\mathrm{p}) = \int_0^T 1_{[lb(\tau), ub(\tau;T)]}(t)\, w(t)\, w^*(t-\tau)$$
$$e^{\frac{2\pi j m t}{T_\mathrm{p}}} dt, \quad -T \leq \tau \leq T \quad (15)$$

and $lb(\tau) = \max(0, \tau)$ and $ub(\tau; T) = \min(T, T + \tau)$. For specific window choices, the integral $I(\tau; m, T, T_\mathrm{p})$ can be calculated analytically. However, in the case of e.g. a Tukey window, six different cases need to be considered and in each case the integral needs presumably to be split in three parts. That is pretty laborious and it is an open question whether this analytical form will lead to faster computations. Instead, analogously to the FT, we will compute the integral numerically as explained in the next section. Finally, note that the variable change can also be implemented on the ACF expression without window.

IV. Computational Aspects

In [2] we have computed the generalized MTSFM signal model by simply point-evaluating the (finite) Fourier sum in (6). Also, we have computed the FT and the ACF of the generalized MTSFM signal model by simple numerical integration schemes (Riemann sums). For the evaluation of this signal model, the Jacobi-Anger expansion is not a simplification, since it requires the approximation of a complex Fourier series, where each term consists of a GBF represented by an integral on $[-\pi, \pi]$ and an exponent. The integrand of the "fundamental mode", i.e., the GBF $J_0(\boldsymbol{b}; j\boldsymbol{a})$, exhibits a MTSFM or Fourier sum, which has the same complexity as the original signal model itself. All other operations necessary to compute the Jacobi-Anger expansion with GBFs make the complexity of that expansion higher than that of the original signal model. However, the GBF represented signal model may still provide insight in e.g. which are the dominant modes or fundamental frequencies.

For the numerical evaluation of the FT and the ACF, the Jacobi-Anger expansion could be a simplification, since the exponents and sinc functions in their expressions can be rapidly evaluated and the GBFs need to be computed only once to evaluate the signal model, its FT, and its ACF. Also, computing the GBFs $J_n(\boldsymbol{b}; j\boldsymbol{a})$, we need to calculate the MTSFM or Fourier sum in their expressions, i.e., the integrand of $J_0(\boldsymbol{b}; j\boldsymbol{a})$, only once since it does not depend on the index n. A computation scheme of the GBFs with integral approximation by Riemann sums reads as follows:

1) Evaluate the (MTSFM or Fourier sum in the) integrand of $J_0(\boldsymbol{b}; j\boldsymbol{a})$ in $2N_\mathrm{int}$ points, where N_int is the number of integration points on an interval of length π.
2) Evaluate the complex exponential $e^{jn\theta}$ at $2N_\mathrm{int}$ points with $n = 1, ..., N_\mathrm{sum}$, where $\pm N_\mathrm{sum}$ is the truncation index of the (infinite) complex Fourier series.
3) Multiply the evaluated integrand of $J_0(\boldsymbol{b}; j\boldsymbol{a})$ and the complex exponential $e^{jn\theta}$ at $2N_\mathrm{int}$ points with $n = 1, ..., N_\mathrm{sum}$. Note that the product of the integrand of $J_0(\boldsymbol{b}; j\boldsymbol{a})$ and the complex exponential with $n = 0$ is trivial.
4) Sum $N_\mathrm{sum} + 1$ times $2N_\mathrm{int}$ products, which yields $J_n(\boldsymbol{b}; j\boldsymbol{a})$ for $n = 0, ..., N_\mathrm{sum}$
5) Evaluate $J_{-n}(\boldsymbol{b}; j\boldsymbol{a}) = J_n^*(-\boldsymbol{b}; -j\boldsymbol{a})$ for $n = 1, ..., N_\mathrm{sum}$.

The second to fourth steps can also be carried out by a single discrete Fourier transform (DFT) applied to a sequence of $2N_\mathrm{int}$ evaluations of the integrand of $J_0(\boldsymbol{b}; j\boldsymbol{a})$. The result is $J_n(\boldsymbol{b}; j\boldsymbol{a})$ for $n = 0, ..., 2N_\mathrm{int} - 1$. Hence, N_sum is maximally $2N_\mathrm{int} - 1$. The DFT can be handled by a fast Fourier transform (FFT). Having computed the GBFs, we can evaluate the signal model (7) (in N_time points), its FT (9) (in N_freq points), and the ACF (11) (in $2N_\mathrm{time}$ points).

In Matlab or Python, an efficient way of computing the FT seems to be looping over the index n in (12) and representing the discretized integrand as a $N_\mathrm{f} \times N_\mathrm{t}$ matrix, where we can sum at once over the second dimension to compute all integrals. This approach may be accelerated by writing the FT as a DFT that can be handled by a FFT, but whether that is more efficient is questionable. It requires $N_\mathrm{f} = N_\mathrm{t}$, while $N_\mathrm{f} \gg N_\mathrm{t}$ if one wants to see the frequency response over a relatively wide frequency range.

The computation of the ACF in (14) can be accomplished as follows. Suppose that we truncate the original loops over n and n' at $\pm N_\mathrm{sum}$. Then, with the applied substitution $m = n - n'$, we have $-2N_\mathrm{sum} \leq m \leq 2N_\mathrm{sum}$ and $-N_\mathrm{sum} \leq n' = n - m \leq N_\mathrm{sum}$. Thus, $-N_\mathrm{sum} + m \leq n \leq N_\mathrm{sum} + m$, but also $-N_\mathrm{sum} \leq n \leq N_\mathrm{sum}$, which implies $\max(-N_\mathrm{sum}, -N_\mathrm{sum} + m) \leq n \leq \min(N_\mathrm{sum}, N_\mathrm{sum} + m)$. Consequently,

$$R(\tau) = \sum_{m=-2N_\mathrm{sum}}^{2N_\mathrm{sum}} I(\tau; m, T, T_\mathrm{p})$$
$$\sum_{n=\max(-N_\mathrm{sum}, -N_\mathrm{sum}+m)}^{\min(N_\mathrm{sum}, N_\mathrm{sum}+m)} J_n(\boldsymbol{b}; j\boldsymbol{a}) J_{n-m}^*(\boldsymbol{b}; j\boldsymbol{a}) e^{\frac{2\pi j(n-m)\tau}{T_\mathrm{p}}}$$
$$(16)$$

We compute the integral I by representing the discretized integrand as a $N_\tau \times N_\mathrm{t}$ matrix. Then, analogously to the FT, we can sum, in Matlab or Python, at once over the second matrix dimension to compute all integrals. As in the case of the FT, it is questionable whether writing the integral as a DFT that can be handled by a FFT is more efficient, since it requires $N_t = N_\tau$, while one may compute with $N_t = N_\tau/2$, because of the ratio of two between the t and τ intervals.

V. Numerical Results

To investigate numerically the complex Fourier expansions of generalized MTSFM signals, we start by considering the first representation case in [2, Sec. III.A], which describes the application of the generalized MTSFM signal model to a classical MTSFM example in [1]. Fig. 1 shows the real signal $\mathrm{Re}(g(t))$ versus time t calculated by four different expressions. The first two curves correspond to the calculation of the signal $\mathrm{Re}(g(t))$ from its original exponential expression. Here, the first curve corresponds to Hague's classical MTSFM with period $T_p = T$ and $Q = 32$ sine functions in the phase expansion [2, Eq. (11)], and the second curve corresponds to the generalized MTSFM with period $T_p = 2T$ and $Q = 64$ cosine and sine functions, where the $2Q+1 = 129$ expansion coefficients are obtained by running the numerical method detailed at the end of [2, Sec. II]. The third and fourth curves are the two complex Fourier series that correspond to the first two cases. In these series we truncate the summation at $n = \pm N_{\mathrm{sum}} = \pm 300$ and we calculate the GBF integrals on $[-\pi, \pi]$ by Riemann sums with $N_{\mathrm{int}} = 600$ integration points per interval of length π, thus 1200 points in total.

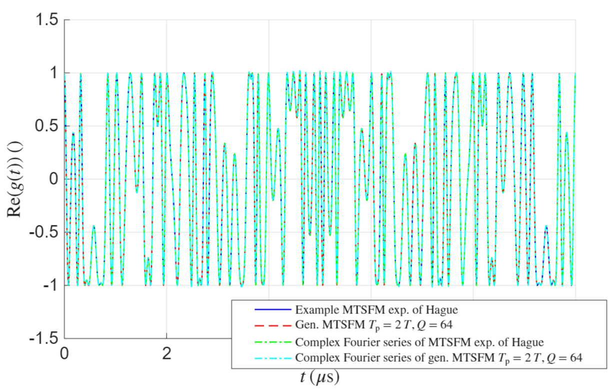

Fig. 1. The signal $\mathrm{Re}(g(t))$ of the example in [2, Fig. 1] versus time t for $N_{\mathrm{sum}} = 300$ and $N_{\mathrm{int}} = 600$.

Fig. 1 shows that the two complex Fourier series approximate accurately the directly evaluated signals. Considering the magnitudes of the expansion coefficients in Fig. 2 calculated for $N_{\mathrm{sum}} = 1200$, $N_{\mathrm{int}} = 2400$, we observe that they decay exponentially until reaching machine accuracy at, approximately, indices ± 500 for the case $T_p = T$ and $Q = 32$, and at approximately indices ± 1000 for the case $T_p = 2T$ and $Q = 64$, in which the generalized MTSFM coefficients are determined by the numerical procedure with the Gram matrix [2, p. 5304]. The exponential decay is in agreement with the theoretical statement in [11, below Eq. (15)]. With respect to the slower convergent Fourier series of the second case, we recall from [2, Sec. III.A] that, in the classical MTSFM example of Hague, the period T yields an infinitely many times continuously differentiable function on the real line. In contrast, for the period $2T$, we chose as coefficients those obtained from the mentioned numerical procedure. In that case, the generalized MTSFM approximates a function which equals the classical MTSFM on $[0, T]$ and zero on $[-T, 0]$. This function is only piecewise differentiable on the real line, since the derivative is undefined in points that are multiples of T. This less smooth behaviour results in a slower convergent Fourier series.

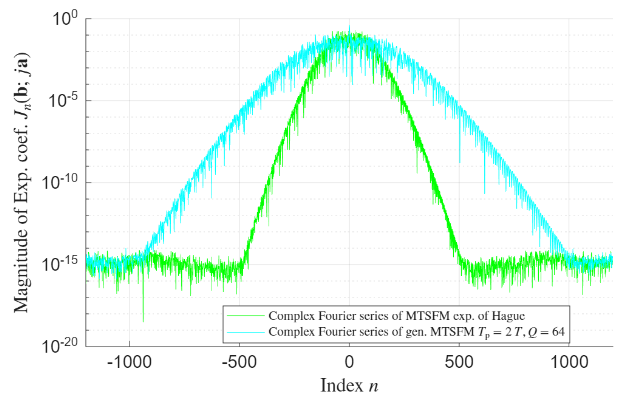

Fig. 2. Magnitudes of the expansion coefficients corresponding to the example in Fig. 1 for $N_{\mathrm{sum}} = 1200$ and $N_{\mathrm{int}} = 2400$.

Despite the nice visualization of the exponential decay of the expansion coefficients of the Jacobi-Anger expansion, their behaviour seems to provide few insight. Hence, the added value of the Jacobi-Anger expansion should come from the (numerical) analysis of the signal's FT and ACF. Fig. 3 shows the FTs for $N_{\mathrm{sum}} = 400$, $N_{\mathrm{int}} = 800$. For the Jacobi-Anger expansion of the example MTSFM of Hague, which has period $T_{\mathrm{period}} = T$, the result seems converged when comparing it to the direct evaluations in Fig. [2, Fig. 2(b)], where 2000 integration points in time are used and where the FT is evaluated for 8001 frequency points. This result has been validated against [1, Fig. 1]. However, even for $N_{\mathrm{sum}} = 400$, thus 801 terms to sum, convergence is still not settled for the case $T_{\mathrm{period}} = 2T$. Between ± 20 MHz and ± 30 MHz the spectrum of this case does not match yet the spectrum of direct integration of the FT. Of course the improved results are obtained at considerable higher computational cost; $N_{\mathrm{sum}} = 400$ means summing 801 terms. In this respect, we can say that the Jacobi-Anger expansion of the FT may only be competitive with the direct integration of the FT in case no window is present (uniform window), since then we only need to sum relatively elementary terms containing a sinc, an exponent, and a GBF coefficient, which we assume computed already for signal evaluation.

The Jacobi-Anger expansion of the ACF converges faster, since two GBF coefficients with exponential decay are multiplied. Fig. 4 shows the results for $N_{\mathrm{sum}} = 200$ and $N_{\mathrm{int}} = 400$, together with the direct integration of the ACF

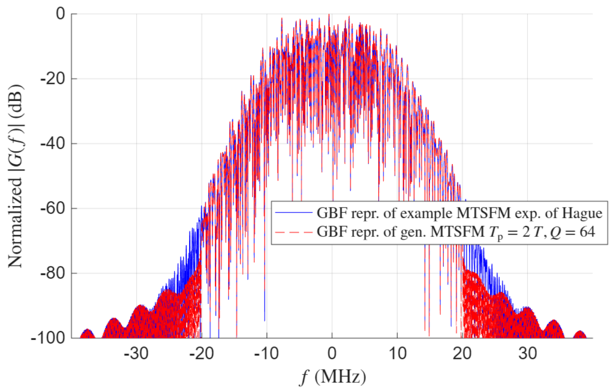

Fig. 3. Magnitudes of the FTs $G(f)$ of the signal $g(t)$ corresponding to Fig. 1 subject to a Tukey window with shape parameter 0.05. The FTs are obtained from the windowed $G(f)$ expression of the Jacobi-Anger expansion of $g(t)$, where $N_{\rm sum} = 400$ and $N_{\rm int} = 800$.

integral of the example MTSFM expansion of Hague with period $T_{\rm p} = T$ and $Q = 32$.[1] Both Jacobi-Anger expansions match the direct-integration result. This nice matching comes however at the cost of significantly more computational effort than direct integration, while we have not been able to identify additional insight in the (generalized) MTSFM's behaviour.

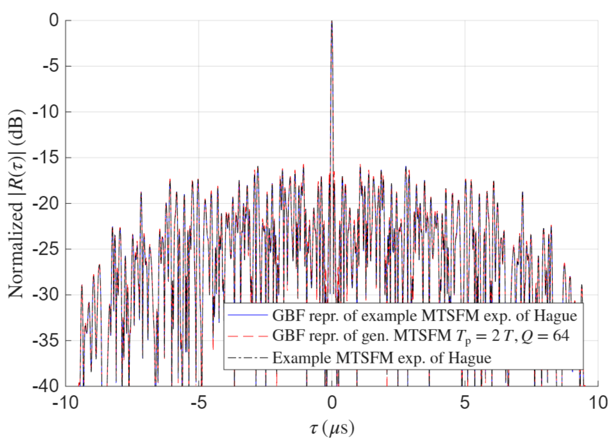

Fig. 4. Magnitudes of the ACFs $R(\tau)$ of the signal $g(t)$ corresponding to Fig. 1 subject to a Tukey window with shape parameter 0.05. The ACFs are obtained from the windowed $R(\tau)$ expression of the Jacobi-Anger expansion of $g(t)$, where $N_{\rm sum} = 200$ and $N_{\rm int} = 400$. Note that black dashed-dotted curve is the result for direct integration of the ACF integral of the example MTSFM expansion of Hague with period $T_{\rm p} = T$ and $Q = 32$.

VI. Conclusions and Future Work

In this paper we have derived Jacobi-Anger expansions for the generalized/extended MTSFM signal model and the corresponding FT and ACF, both without and with window. Also, we detailed how to, analytically and numerically, evaluate the GBF expansion coefficients and the FT and ACF expansions. We applied our theory to the first representation example of our previous paper [2], in which a classical example MTSFM is handled by the generalized MTSFM. The two numerically evaluated Jacobi-Anger expansions of this example match our earlier results obtained with direct integration of the FT and ACF integrals, which in turn have been validated with the original reference of the classical example MTSFM. We have not yet been able to identify a clear advantage of using the Jacobi-Anger expansions instead of directly evaluating the signal and its corresponding FT and ACF, nor in terms of (numerical) computation, neither in terms of additional insight in the behaviour of these quantities. To shed more light on the great simplification of the analysis of the MTSFM model by Jacobi-Anger expansions mentioned in [1, p. 1278], our next focus, before end August 2026, will be on considering the other representation examples in our previous paper and, in particular, using the Jacobi-Anger expansions in optimization. To this end, we will derive Jacobi-Anger expansions of the RMS bandwidth and the integrated sidelobe ratio (ISR), and their derivatives. Subsequently, we will repeat the optimization examples of our previous paper with Jacobi-Anger expansions instead of the original generalized MTSFM signal model.

[1] Recall from [2, Fig. 2(a)] that the direct integration of the ACF integral of the generalized MTSFM expansion with period $T_{\rm p} = 2T$ and $Q = 64$ mimics the result of the example MTSFM expansion of Hague on $[0, T]$.

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