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Application of the Rosenblatt transformation in First-Order System Reliability approximations

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ABSTRACT

The reliability assessment of structural systems presents a significant challenge in structural engineering. A commonly employed approximation is the First-Order System Reliability Method (FOSRM), which estimates system reliability using the FORM component reliabilities and sensitivity factors. An essential step in FORM involves transforming the random vector $\mathbf{X}$ into the standard vector $\mathbf{U}$, often using the Rosenblatt transformation (RT). Several studies demonstrated that different conditioning orders in the RT yield different FORM component results. This study investigates how these differences on component level propagate into the FOSRM system level. We conducted several typical engineering case studies with various failure probabilities, system sizes, and dependency structures (Gaussian and Frank Copula). For the Frank Copula, different Rosenblatt conditioning orders systematically yielded different FOSRM results, with most cases showing differences between 10% and 30% in estimated failure probability. For some systems, these differences increased with system size, suggesting that greater variations might be observed for larger systems. Notably, systems with Gaussian Copula functions also proved vulnerable to the Rosenblatt conditioning order when different components were assessed with different conditioning orders. The observed differences were larger than previously reported and should be carefully considered in uniform safety assessments.

1. Introduction

1.1. Context

Calculating the reliability of complex systems is a challenging problem in structural reliability. This especially holds for systems with many failure modes and statistically dependent random variables. The starting point of any system reliability problem is the definition of an idealized system limit state function $Z_{\text{sys}} = g_{\text{sys}}(\mathbf{X})$ which describes the boundary between the safe ($Z_{\text{sys}} \geq 0$) and failure ($Z_{\text{sys}} < 0$) domain, and where $\mathbf{X} = \{X_1, \ldots, X_k, \ldots, X_n\}$ represents the vector of basic random variables. For relatively simple systems, the system limit state function (LSF) can be described by exhaustive physical or empirical models that meticulously describe all interactions between the (structural) elements, e.g., non-linear finite element models. This approach is often impossible for complex systems due to the different physical nature, and therefore modelling strategies, of the separate elements involved. Consider, for example, a flood defence system with structural, geotechnical, mechanical, and electrical elements; each of these elements is modelled by different experts and modelling programs, and treated with their own, individual LSFs. In those situations, the system is simplified into separate, non-interacting failure modes with their own LSFs $\{Z_1, \ldots, Z_i, \ldots, Z_m\}$, from here on referred to as 'component LSFs' and m the number of components. The system LSF is obtained as a logical function of the component LSFs. The system failure probability $\mathrm{P}_{\mathrm{f,sys}}$ is defined by:

$$\mathrm{P}_{\mathrm{f,sys}} = \mathrm{P}(Z_{\text{sys}} < 0) = \int_{Z_{\text{sys}}<0} f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x} \quad (1)$$

where $f_{\mathbf{X}}(\mathbf{x})$ represents the n-dimensional joint probability density function (pdf) of the basic random variables $\mathbf{X}$.

The modelling of the joint pdf $f_{\mathbf{X}}(\mathbf{x})$ plays a crucial role in the outcomes of the system reliability problem. It is typically modelled as a combination of experimental data and expert knowledge, and can be written as a set of marginal pdfs $\{f_{X_k}(x_k)\}$ and a dependency structure. For system reliability problems, this dependency structure is often high-dimensional and complex since it includes both within-component and between-component dependencies.

Several dependency structures are employed in engineering problems, of which the Pearson correlation matrix is probably the most

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widely employed. It assumes bi-variate linear dependencies between pairs of basic random variables [1]. Lebrun and Dutfoy [2] showed that the application of the Pearson correlation matrix combined with the Nataf transformation [3] implicitly assumes a Gaussian Copula function. In many engineering problems, however, the Gaussian Copula function is an incorrect representation of reality — leading to significant errors in the reliability calculations [4–6]. The application of non-Gaussian Copula functions in engineering problems is therefore increasing, such as in wind engineering [7], hydraulic engineering [8–10], geotechnical engineering [4] or system engineering [11–13].

1.2. Problem definition

The reliability calculation of engineering systems is often challenging due to the involvement of many statistically dependent component LSFs and random variables. While Crude Monte Carlo (CMC) methods could theoretically be applied in such cases, they are often impractical due to: (i) the combination of low failure probabilities and time-consuming LSFs or; (ii) the fact that different experts treat different component LSFs, which later need to be combined into a system reliability.

In 1982, Hohenbichler and Rackwitz [14] introduced a first-order approximation to the system reliability problem which overcomes both of these problems and is still thoroughly employed today, see e.g. [15–18]. The method, from here on referred to as the First-Order System Reliability Method (FOSRM), relies on the first-order component reliability calculations (FORM) [19] and the subsequent combination of the FORM results into the system reliability.

A crucial step in FORM regards the iso-probabilistic transformation of the random vector $\mathbf{X}$ into the independent, standard normal vector $\mathbf{U}$ [20]. Several iso-probabilistic methods exist for this purpose, of which the applicability depends on the dependency structure involved in the system. For the Pearson correlation matrix, typically the classical Nataf transformation is employed [3,21]. For elliptical [22] and Archimedean Copula functions [23] generalizations of the Nataf transformation were proposed. The most general iso-probabilistic method is the Rosenblatt transformation [24], which can be applied to all Copula types (Gaussian, elliptical, Archimedean, etc.).

The essence of the Rosenblatt transformation is the conditioning of the random variables upon each other, for which in total $n!$ possible conditioning orders exist [24]. From a mathematical perspective, all conditioning orders are equivalent and equally valid. Several studies have shown that for non-Gaussian Copula functions, different conditioning orders yield different FORM results, i.e. reliability indices and sensitivity factors [25–27]. These studies primarily focus on linear LSFs and component-level analysis, leaving the potential impact on typical engineering systems with numerous components and non-linear LSFs unexplored. It is also not well understood what happens when different components are assessed with different conditioning orders before aggregating them into the FOSRM system reliability - a scenario that might occur when different experts assess different components.

This knowledge gap often results in the specifics of the conditioning order being neglected or overlooked in engineering practice, potentially causing ambiguity in the FOSRM reliability outcomes. Since decisions regarding the safety of structures often hinge on FOSRM system reliability assessments [28], it is crucial to better understand the potential impact of the conditioning order on a system level.

1.3. Aim of this study

This study investigates the impact of the Rosenblatt conditioning order on FOSRM system reliability assessments. It is hypothesized that the impact of the conditioning order at the (FOSRM) system level differs from the impact on the (FORM) component level, and may become more pronounced as the system size increases. Additionally, it is hypothesized that using different conditioning orders for different components could result in significant discrepancies in (FOSRM) system reliability outcomes, even when Gaussian Copula functions are employed. To address these hypotheses, we will explore the following research questions:

- Does the choice of the conditioning order only affect systems with non-Gaussian Copula functions, or also those with (solely) Gaussian Copula functions?
- What order of magnitude is the difference in FOSRM results when one conditioning order is applied as compared to the other?
- Does the impact of the conditioning order systematically increase with an increasing number of system components?
- What happens when different system components are assessed with different conditioning orders?

It is hereby noted that practical engineering applications encompass a vast array of structural systems, each with varying LSFs, system sizes, marginal distributions, and dependency structures. The answers to the research questions therefore will vary from system to system, making it difficult to provide generalized statements. The primary aim of this paper is therefore to deepen our understanding of the issue and to bring awareness to engineering practice; it does not offer definitive solutions or universal truths for each of the research questions.

1.4. Approach

We will address the research questions through two academic examples with different purposes. In the first example, we will demonstrate how both Gaussian and non-Gaussian (Frank) Copula functions are affected by the application of different conditioning orders. To do this, we will use a well-documented example from the literature, originally conducted at the element level but now extended to the system level. In this example, we will go through the FOSRM calculation procedure step by step, focusing primarily on the qualitative understanding of the intermediate results.

In the second example, we will numerically investigate the impact of the Rosenblatt conditioning order for a variety of typical engineering systems. The systems include non-linear LSFs, non-linear dependency structures, and a range of typical failure probabilities (between 10^{-2} and 10^{-5}) and system sizes (between 2 and 10). The example is implemented as a parametric study and aims to gain a quantitative feeling of the order of magnitude of the differences in FOSRM caused by the conditioning order. The differences between the conditioning orders are quantified through impact factors measuring the absolute and relative differences.

Both examples comprise series systems with identical components, as this is representative of many engineering problems, such as flood defence systems [15] or corroded pipelines [29]. We limit ourselves to the situation where each system component includes two statistically dependent random variables, which are connected via a Gaussian or Frank Copula function (see Table 1). The Frank Copula provides greater flexibility in modelling dependence structures compared to the Gaussian Copula, and is often a candidate in engineering applications, such as for the modelling of peak discharges and volumes [30], rainfall and tide [31], and wave heights and wind speeds [32]. Additionally, the Frank Copula has a closed-form solution to the Rosenblatt transformation, which simplifies the calculations.

1.5. Reading guide

The subsequent sections are organized as follows. Section 2 presents the methodology and theoretical framework as adopted in this study, including the details of the FOSRM implementations (Section 2.1), the Rosenblatt transformation (Section 2.2), and the impact factors (Section 2.3). Next, we present the results and calculations of the two examples in Sections 3 and 4. Based on the results, we discuss the research questions in Section 5. The study closes with some conclusions and recommendations in Section 6.

Table 1
Parametrization of bi-variate Copula functions as applied in this study. The parameters $\{p_1, p_2\}$ represent standard uniform variables.

Copula type	Copula function	Range input parameter
Gaussian	$\Phi_2(\Phi^{-1}(p_1), \Phi^{-1}(p_2); \mathfrak{R})$	$\mathfrak{R} = \{\rho_{ij}\}$ with $-1 < \rho_{ij} \leq 1$
Frank	$\frac{1}{\theta}\ln\left(1 + \frac{(\exp(-\theta p_1)-1)(\exp(-\theta p_2)-1)}{\exp(-\theta)-1}\right)$	$\theta \in (-\infty, \infty)/\{0\}$

2. Theoretical framework

2.1. First-order system reliability method (FOSRM)

Given a series system consisting of m non-interacting components with limit state functions $\{Z_i = g_i(\mathbf{X})\}$, and the n-dimensional vector of basic random variables $\mathbf{X} = \{X_1, \dots, X_k, \dots, X_n\}$. The FOSRM failure probability is calculated using the steps below.

2.1.1. FOSRM-step 1: FORM component approximations

The first step in FOSRM regards the FORM approximation of the component reliabilities [19,20]. Given a component with LSF $Z_i = g_i(\mathbf{X})$. The essence of FORM lies in the iso-probabilistic transformation of the basic random variables from the dependent, non-normal $\mathbf{X}$-space to the independent, standard normal $\mathbf{U}$-space, for which we employ the Rosenblatt transformation (see Section 2.2). The next step is to linearize the transformed LSF using the first-order Taylor expansion, after which it is sought for the design point $\mathbf{u}^*$ on the LSF with the highest probability density. The distance between $\mathbf{u}^*$ and the origin is referred to as the Hasofer–Lind reliability index [19] and represents the component reliability index $\beta^{(i)}_{\text{comp}}$, where i indicates the given component. The direction cosines in $\mathbf{u}^*$ represent the sensitivity factors, captured in the vector $\boldsymbol{\alpha}^{(i)} = \{\alpha^{(i)}_1, \dots, \alpha^{(i)}_k, \dots, \alpha^{(i)}_n\}$, where i denotes the component and (k) the random variable. For an m-component system with n random variables, this leads to an m-dimensional vector of component reliability indices $\boldsymbol{\beta}_{\text{comp}} = \{\beta^{(1)}_{\text{comp}}, \dots, \beta^{(i)}_{\text{comp}}, \dots, \beta^{(m)}_{\text{comp}}\}$ and an $m \times n$-dimensional matrix of sensitivity factors $\boldsymbol{\alpha} = \{\alpha^{(i)}_k\}$. Under certain conditions (linear LSF and independent, normally distributed random variables) the FORM reliability results are exact. For all other situations, the FORM results are an approximation.

In this study, the FORM calculations are conducted using the Hasofer–Lind–Rackwitz–Fiessler algorithm [19,33]. For sufficient accuracy, the proximity of the design point to the LSF (quantified by $\epsilon_Z \leq Z/\sigma_Z < \epsilon_Z$) is set to $\epsilon_Z = 0.0001$, and the accuracy of the gradient pointing towards the origin (quantified by $(1-\epsilon_\beta)(\mathbf{u}_0^{*T}\mathbf{u}_0^*) \leq \beta^2 < (1+\epsilon_\beta)(\mathbf{u}_0^{*T}\mathbf{u}_0^*)$) is set to $\epsilon_\beta = 0.0001$.

2.1.2. FOSRM-step 2: Estimation of between-component dependency

The second step in FOSRM is the estimation of the statistical dependency between all components $Z_1, \dots, Z_i, \dots, Z_m$. For linear LSFs, Gaussian random variables, and Pearson correlation between all variables, the between-component dependency is exactly described by the $m \times m$ Pearson correlation matrix $\mathfrak{R}_Z = \{\rho^{(ij)}\}$ [14]. The entries of the Pearson correlation matrix $\mathfrak{R}_Z$ are determined as the inner product of the FORM sensitivity factors associated with the ith and jth component:

$$\rho^{(ij)} = \sum_{k=1}^{n} \alpha^{(i)}_k \alpha^{(j)}_k = \boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{\alpha}^{(j)} \tag{2}$$

Practical engineering problems rarely include linear LSFs, normally distributed random variables, or Pearson correlation between all random variables. The application of the Pearson correlation matrix will thus often be inadequate. To motivate its application nonetheless, it is typically assumed that the strengths of the calculated correlation coefficients $\{\rho^{(ij)}\}$ provide a sufficiently accurate representation of the between-component dependencies around the design points, since they follow from the sensitivity matrix $\boldsymbol{\alpha}$ as obtained in FOSRM-step 1.

Roscoe's adaptation. To evaluate Eq. (2), FOSRM requires that in each of the m FORM calculations, all n random variables are included; this also holds for random variables in $\mathbf{X}$ that are not represented in the given component LSF Z_i. For complex systems, however, the number of dependent random variables and system components can be extremely large (say $n = 1000$ random variables and $m = 500$ components or more for a typical flood defence system). Calculating the sensitivity factors for all n random variables in all m components may lead to relatively long calculation times, especially when the finite difference method is employed [16]. Similarly, conducting the n-dimensional Rosenblatt transformation in each FORM iteration may lead to relatively long calculation times — especially when no closed-form solution is available and numerical integration is required. In those cases, FOSRM will lose its numerical advantages as compared to CMC. For systems comprising components with (sets of) identical random variables, therefore, Roscoe et al. [15] present a more efficient way to calculate the pairwise correlation coefficients $\{\rho_{ij}\}$. These systems can be written as:

$$\begin{aligned} Z_1 &= g_1\left(X^{(1)}_1, \dots, X^{(1)}_k \dots, X^{(1)}_{n_{\text{set}}}\right) \\ &\vdots \\ Z_i &= g_i\left(X^{(i)}_1, \dots, X^{(i)}_k \dots, X^{(i)}_{n_{\text{set}}}\right) \\ &\vdots \\ Z_m &= g_m\left(X^{(m)}_1, \dots, X^{(m)}_k \dots, X^{(m)}_{n_{\text{set}}}\right) \end{aligned} \tag{3}$$

where n_{set} represents the total number of variable-types within a set (or within each component), leading to a total number of $n = n_{\text{set}} \cdot m$ random variables within the entire system. The systems have the following requirements:

(i) The variables $\{X^{(i)}_k, X^{(j)}_k\}$ are different, yet identically distributed random variables. The exact formulation of the LSFs may differ from component to component, as long as the same set of random variables is involved.

(ii) The statistical dependency between the identically distributed random variables $\{X^{(i)}_k, X^{(j)}_k\}$ is described by the Gaussian Copula function with strength parameter $\rho^{(ij)}_{\text{auto},k}$, referred to as the autocorrelation. The statistical dependency between random variables within the same component $\{X^{(i)}_k, X^{(i)}_{\sim k}\}$ can be of any kind (Gaussian or non-Gaussian Copula function).

(iii) There exists no statistical dependency between non-identical variables from different components $\{X^{(i)}_k, X^{(j)}_{\sim k}\}$ for all i, j, k.

For systems fulfilling all three requirements, Roscoe et al. [15] provide the following expression to determine the between-component correlation:

$$\rho^{(ij)} = \sum_{i=1}^{n_{\text{set}}} \alpha^{(i)}_k \alpha^{(j)}_k \rho^{(ij)}_{\text{auto},k} \tag{4}$$

In contrast to Eq. (2), the evaluation of Eq. (4) does not require each component FORM calculation to include all n random variables from the entire system. Instead, each component FORM calculation only needs to include the n_{set} random variables in the given component LSF, which is typically much less.

It is important to remark that Roscoe's adaptation can only be applied to systems fulfilling all three requirements (i), (ii) and (iii). Hereby the first two requirements (i) and (ii) are strict prerequisites, whereas the last requirement (iii) is optional; when it is fulfilled, expression (4) yields the same results as expression (2) and if not, it yields an approximation. Although requirement (iii) is not always fulfilled in reality, Roscoe's adaptation is often the first choice for large-scale system reliability problems nonetheless, such as in the probabilistic assessment of Dutch flood defence systems [28]. It is therefore employed in this study as well.

2.1.3. FOSRM-step 3: Calculation of system failure probability

The last step in FOSRM is to calculate the (series) system failure probability via:

$$\mathrm{P}_{\mathrm{f,sys}} = 1 - \Phi_m(\boldsymbol{\beta}_{\mathrm{comp}}, \mathfrak{R}_{\mathrm{Z}}) \tag{5}$$

where $\Phi_m(.)$ represents the m-dimensional standard normal cumulative distribution (cdf), and $\boldsymbol{\beta}_{\mathrm{comp}}$ and $\mathfrak{R}_{\mathrm{Z}}$ as determined in FOSRM-step 1 and 2 respectively. No exact solution exists for Eq. (5). Depending on the characteristics of $\mathfrak{R}_{\mathrm{Z}}$, several simplified formulas can be derived for the evaluation of Eq. (5). For fully-independent systems with $\rho^{(ij)} = 0$ for all i, j, Eq. (5) can be simplified to:

$$\mathrm{P}_{\mathrm{f,sys}} = \sum_{i=1}^{m} \mathrm{P}^{(i)}_{\mathrm{f,comp}} - \prod_{i=1}^{m} \mathrm{P}^{(i)}_{\mathrm{f,comp}} \tag{6}$$

where $\mathrm{P}^{(i)}_{\mathrm{f,comp}} = \Phi(-\beta^{(i)}_{\mathrm{comp}})$ represents the equivalent component failure probability as calculated with FORM. For fully dependent systems with $\rho^{(ij)} = 1$ for all i, j, Eq. (5) can essentially be simplified to:

$$\mathrm{P}_{\mathrm{f,sys}} = \min\left[\mathrm{P}^{(1)}_{\mathrm{f,comp}}, \dots, \mathrm{P}^{(i)}_{\mathrm{f,comp}}, \dots, \mathrm{P}^{(m)}_{\mathrm{f,comp}}\right] \tag{7}$$

For equi-correlated systems with $\rho^{(ij)} = \rho$ for all i, j, Eq. (5) can be simplified to the one-dimensional integral from Dunnet and Sobol [34]:

$$\int_{-\infty}^{\infty} \phi(t) \prod_{i=1}^{m} \Phi\left(\frac{\beta^{(i)}_{\mathrm{comp}} - \sqrt{\rho}t}{\sqrt{1-\rho}}\right) \mathrm{d}t \tag{8}$$

where $\phi(.)$ is the pdf of the standard normal distribution. For all other cases, we employed the mvncdf-function as implemented in MATLAB [35] which is based on the algorithms addressed in [36–41]. Two requirements of the implementation are that: (1) off-diagonal correlation entries cannot be equal to $\rho^{(ij)} \neq 1$ and that; (2) the resulting correlation matrix $\mathfrak{R}_{\mathrm{Z}}$ is symmetric positive definite (SPD). In the case we had an off-diagonal entry equal to $\rho^{(ij)} = 1$, we, therefore, decreased the entry with 10^{-9}. When we dealt with a non-SPD matrix we sought for the 'nearest SPD matrix' using the implementation in [42]. Neither of the approximations was found to have a significant influence on the results.

2.2. Rosenblatt iso-probabilistic transformation

Formal definition. Let $\mathbf{X} = \{X_1, X_2\}$ be the random dependent vector with marginal distribution functions $\{F_k(x_k)\}$ and joint distribution $F_{\mathbf{X}}(\mathbf{x})$. The Rosenblatt transformation T^{Ros} of the vector $\mathbf{X}$ into a vector of independent, standard normal vector $\mathbf{U} = \{U_1, U_2\}$ is given by [27]:

$$\mathbf{U} = T^{\mathrm{Ros}}(\mathbf{X}) = T_2^{\mathrm{Ros}} \circ T_1^{\mathrm{Ros}}(\mathbf{X}) \tag{9}$$

where $T_1^{\mathrm{Ros}} : \mathbb{R}^n \to \mathbb{R}^n$ transforms random vector $\mathbf{X}$ to the random Uniform vector $\mathbf{Y} = \{Y_1, Y_2\}$ over $[0,1]^n$ with independent Copula:

$$\begin{aligned} y_1 &= F_1(x_1) \\ y_2 &= F_2(x_2|x_1) \end{aligned} \tag{10}$$

where $F_2(x_2|x_1)$ represents the conditional cumulative distribution function of X_2 given the variable X_1, which is often evaluated numerically due to the lack of analytical solutions. For the conditioning, in total $n!$ conditioning orders are available, which is limited to $n! = 2$ in the bi-variate case.

The next step $T_2^{\mathrm{Ros}} : \mathbb{R}^n \to \mathbb{R}^n$ transforms the independent Uniform random vector $\mathbf{Y}$ to the independent standard Normal vector $\mathbf{U}$ with zero mean and unit variance:

$$\begin{aligned} u_1 &= \Phi^{-1}(y_1) \\ u_2 &= \Phi^{-1}(y_2) \end{aligned} \tag{11}$$

where $\Phi^{-1}(.)$ represents the inverse of the standard normal distribution.

Investigated conditioning orders. Given that this study limits to bi-variate dependencies only, each component can be assessed with $n! = 2$ conditioning orders. We define these conditioning orders as:

- *Canonical conditioning order*, which yields from the (coincidental) order given to the random vector $\mathbf{X} = \{X_1, X_2\}$, and;
- *Reverse conditioning order*, which is taken as $\mathbf{X} = \{X_2, X_1\}$. Lebrun and Dutfoy [27] refer to this as the 'inverse' conditioning order.

Both conditioning orders are equivalent from a mathematical point of view. Throughout this paper, all FORM and FOSRM results are annotated with the superscripts (C) for the canonical conditioning order, and (R) for the reverse conditioning order.

Uniform and mixed systems. Given a system of m components. Each component in the system can either be assessed with the canonical conditioning order or with the reverse conditioning order. This results in a total of 2^m conditioning cases. In the following sections, we will distinguish between the following systems:

- *Uniform systems:* All components are assessed with the same conditioning order.
  - *Canonical systems:* All components are assessed with the canonical conditioning order.
  - *Reverse systems:* All components are assessed with the reverse conditioning order.
- *Mixed systems:* Different components in the system are assessed with different conditioning orders.

2.3. Impact factors

We quantify the impact of the Rosenblatt conditioning through three (dimensionless) metrics: the absolute impact factor (IF-abs), the relative impact factor (IF-rel), and the CMC impact factor (IF-CMC), see below. The impact factors primarily focus on the differences between uniform systems rather than the mixed systems. Differences as obtained for mixed systems are therefore only discussed qualitatively. The impact factors are defined at the level of failure probabilities rather than reliability indices, due to the non-linearities involved in the latter.

Absolute impact factor (IF-abs). The absolute impact factor compares the absolute differences between the FOSRM results on the system level, with those obtained on component level. It is defined as:

$$\text{IF-abs} = \frac{|\mathrm{P}^{\mathrm{C}}_{\mathrm{f,sys}} - \mathrm{P}^{\mathrm{R}}_{\mathrm{f,sys}}|}{|\mathrm{P}^{\mathrm{C}}_{\mathrm{f,comp}} - \mathrm{P}^{\mathrm{R}}_{\mathrm{f,comp}}|} \tag{12}$$

with $\mathrm{P}^{\mathrm{C}}_{\mathrm{f,sys}}$ and $\mathrm{P}^{\mathrm{R}}_{\mathrm{f,sys}}$ representing the canonical and reverse FOSRM system failure probability and $\mathrm{P}^{\mathrm{C}}_{\mathrm{f,comp}}$ and $\mathrm{P}^{\mathrm{R}}_{\mathrm{f,comp}}$ the canonical and reverse FORM component failure probabilities.

Relative impact factor (IF-rel). The relative impact factor quantifies the magnitude of the differences in conditioning order relative to the failure probability. It is defined as:

$$\text{IF-rel} = \frac{|\mathrm{P}^{\mathrm{C}}_{\mathrm{f,sys}} - \mathrm{P}^{\mathrm{R}}_{\mathrm{f,sys}}|}{\mathrm{P}^{\mathrm{C}}_{\mathrm{f,sys}}} \tag{13}$$

The larger IF-rel, the larger the difference between the canonical and reverse system (and the more careful one should be with randomly adopting a given conditioning order).

CMC impact factor (IF-CMC). The CMC impact factor addresses the differences caused by the Rosenblatt conditioning order, as compared to other simplifications inherent to FOSRM, such as the linearization of the component LSFs or the assumption of the Pearson correlation matrix. It is defined as:

$$\text{IF-CMC} = \frac{|\mathrm{P}^{\mathrm{C}}_{\mathrm{f,sys}} - \mathrm{P}^{\mathrm{R}}_{\mathrm{f,sys}}|}{|\mathrm{P}_{\mathrm{f,sys,CMC}} - \mathrm{P}_{\mathrm{f,sys,FOSRM}}|} \tag{14}$$

with $\mathrm{P}_{\mathrm{f,sys,FOSRM}} = 0.5\left(\mathrm{P}^{\mathrm{C}}_{\mathrm{f,sys}} + \mathrm{P}^{\mathrm{R}}_{\mathrm{f,sys}}\right)$ representing the mean value of the canonical and reverse FOSRM system failure probabilities.

3. Example 1: Comparison Gaussian and non-Gaussian (Frank) Copula

In this example, we assess how both systems with a Gaussian and non-Gaussian (Frank) Copula function are affected by the choice of the Rosenblatt conditioning order in FOSRM calculations. For this purpose, we get inspired from the well-documented example from Lebrun and Dutfoy [2], which was originally conducted at the element level, but now extended to a two-component system level. We will go through the calculation procedure step by step, where we focus on intermediate results and a general understanding. The investigated system represents a series system of two identical components with the following LSFs:

$$\begin{aligned} Z_1 &= g_1(\mathbf{X}) = 8X_1^{(1)} + 2X_2^{(1)} - 1 \\ Z_2 &= g_2(\mathbf{X}) = 8X_1^{(2)} + 2X_2^{(2)} - 1 \end{aligned} \tag{15}$$

The variables $\{X_1^{(1)}, X_1^{(2)}\}$ are identically distributed according to an exponential distribution with parameter $\lambda_1 = 1$ and the variables $\{X_2^{(1)}, X_2^{(2)}\}$ according to an exponential distribution with parameter $\lambda_2 = 3$, hereby fulfilling Roscoe's first (i) requirement.

Similar to Lebrun and Dutfoy [2], the within-component dependencies between $\{X_1^{(i)}, X_2^{(i)}\}$ are modelled by the Gaussian Copula function with Pearson correlation $\rho_{X_1,X_2} = 0.5$ or the Frank Copula with strength parameter $\theta = 4.73$ (Lebrun and Dutfoy chose $\theta = 10$ here). The latter strength parameter $\theta = 4.73$ was chosen such that its application leads to a linear dependency identical to $\rho_{X_1,X_2} = 0.5$ and thus a valid comparison between the Gaussian and Frank Copula is possible.

For the between-component dependencies of $\{X_k^{(1)}, X_k^{(2)}\}$ we adopt a Gaussian Copula function with strength parameter $\rho^{(ij)}_{\mathrm{auto},k} = 1$ (full dependency), or $\rho^{(ij)}_{\mathrm{auto},k} = 0$ (full independency) - in both cases fulfilling Roscoe's second (ii) requirement. Note that for $\rho^{(ij)}_{\mathrm{auto},k} = 1$ Roscoe's third requirement (iii) of zero cross-correlation is not fulfilled, since $\{X_1^{(1)}, X_2^{(2)}\}$ and $\{X_1^{(2)}, X_2^{(1)}\}$ are automatically dependent through the within-component and between-component dependencies. How this impacts the obtained system reliability results is not assessed here.

3.1. FORM component approximations (FOSRM-step 1)

Fig. 1 shows the component LSFs in the standard normal $\mathbf{U}$-space for the LSFs with the Gaussian (left) and Frank (right) Copula functions respectively. The results for the canonical conditioning order are presented in black and the results for the reverse conditioning order are presented in grey.

For the Gaussian system, the FORM component approximation leads to identical failure probabilities ($\mathrm{P}^{\mathrm{C}}_{\mathrm{f,comp}} = \mathrm{P}^{\mathrm{R}}_{\mathrm{f,comp}} = 0.098$), but different sensitivity vectors ($\boldsymbol{\alpha}^{\mathrm{C}} = \{0.992, 0.128\}$ versus $\boldsymbol{\alpha}^{\mathrm{R}} = \{0.795, 0.606\}$). For the Frank system, however, different failure probabilities are found for the different conditioning orders ($\mathrm{P}^{\mathrm{C}}_{\mathrm{f,comp}} = 0.102$ versus $\mathrm{P}^{\mathrm{R}}_{\mathrm{f,comp}} = 0.114$), as well as different sensitivity vectors ($\boldsymbol{\alpha}^{\mathrm{C}} = \{0.996, 0.0916\}$ versus $\boldsymbol{\alpha}^{\mathrm{R}} = \{0.725, 0.688\}$). Lebrun and Dutfoy [27] explain this by the fact that for the Gaussian Copula functions, a change in conditioning order amounts to an additional orthogonal transformation T_3^{Ros} in Eq. (9) which changes the coordinates of the design point $\mathbf{u}^*$ (and thus sensitivity factors), but not its norm (and thus the reliability index). For non-Gaussian Copula functions such as the Frank Copula, however, the LSFs are not linked by an orthogonal transformation, meaning that also the distances from the design point to the origin differ for both conditioning orders.

3.2. Between-component dependency (FOSRM-step 2)

The between-component dependency is obtained by implementing the sensitivity vectors and autocorrelations into Roscoe's expression (4). For both the Gaussian and the Frank system, however, the sensitivity vectors differed for the canonical and reverse conditioning order. This leads to four possible conditioning cases, including the uniform canonical (CC) and reverse (RR) systems, as well as the mixed systems (CR, RC). We now discuss the results for the dependent and independent system separately.

Dependent system. For the uniform systems (CC, RR), the resulting pairwise correlation coefficient yielded equal to $\rho^{(ij)} = 1$ regardless of the applied Copula function. This makes sense, because also in reality there exists a full linear dependency between the two components. For the mixed systems (CR, RC), however, lower correlation coefficients are obtained, with $\rho^{(12)} = 0.866$ for the Gaussian system and $\rho^{(12)} = 0.785$ for the Frank system. For the given example this means that the application of different conditioning orders in different components thus leads to an erroneous and (in this case) lower estimation of the between-component dependency than present in reality.

Independent system. For the independent systems, the substitution of $\rho^{(12)}_{\mathrm{auto},i} = 0$ into Roscoe's expression (4) leads to the adequate value of zero correlation ($\rho^{(ij)} = 0$) for all system cases (CC, RR, CR, RC). This holds for both the Gaussian and the Frank system.

3.3. System failure probability and impact factors (FOSRM-step 3)

Fig. 2 shows the resulting FOSRM and CMC results of the Gaussian (left) and Frank (right) system. The canonical results are presented as black dots, the reverse results as grey dots, and the mixed results as black crosses. We now discuss the results for the dependent and independent system separately.

Dependent system. For the Gaussian system, the FOSRM results of the canonical and reverse systems overlap, with $\mathrm{P}^{\mathrm{CC}}_{\mathrm{f,sys}} = \mathrm{P}^{\mathrm{RR}}_{\mathrm{f,sys}} = 0.098$. The corresponding CMC system failure probability is $\mathrm{P}^{\mathrm{CMC}}_{\mathrm{f,sys}} = 0.0932$. Since there is no difference between the canonical and reverse systems, the relative and CMC impact factors are equal to IF-rel = IF-CMC = 0.

For the Frank system, the FOSRM results for the canonical and reverse systems differ, with $\mathrm{P}^{\mathrm{CC}}_{\mathrm{f,sys}} = 0.102$ for the canonical system and $\mathrm{P}^{\mathrm{RR}}_{\mathrm{f,sys}} = 0.114$ for the reverse system. This leads to the relative impact factor of IF-rel = 0.12, meaning that the choice of the conditioning order affects the obtained FOSRM system failure probability by about 12%. The corresponding CMC system failure probability is $\mathrm{P}^{\mathrm{CMC}}_{\mathrm{f,sys}} = 0.0872$, resulting in an IF-CMC value of 0.81. This value is significant, suggesting that the differences between the uniform systems (CC) and (RR) are approximately equivalent to the general discrepancies inherent to FOSRM.

Focusing on the mixed systems, we see that for both the Gaussian and Frank systems the mixed systems (CR, RC) lead to significantly higher FOSRM failure probabilities than the uniform systems (CC, RR), with $\mathrm{P}^{\mathrm{CR}}_{\mathrm{f,sys}} = \mathrm{P}^{\mathrm{RC}}_{\mathrm{f,sys}} = 0.133$ for the Gaussian system and $\mathrm{P}^{\mathrm{CR}}_{\mathrm{f,sys}} = \mathrm{P}^{\mathrm{RC}}_{\mathrm{f,sys}} = 0.156$ for the Frank system. In both cases, this is caused by the inadequate (too low) estimation of the pairwise correlation coefficient of the between-component dependency, causing the system to behave more 'independently' than it is in reality.

Independent system. The general findings of the independent system resemble those of the dependent system. For the Gaussian system, the uniform systems (CC, RR) again yield the same FOSRM outcomes $\mathrm{P}^{\mathrm{CC}}_{\mathrm{f,sys}} = \mathrm{P}^{\mathrm{RR}}_{\mathrm{f,sys}} = 0.186$ and thus IF-rel = IF-CMC = 0. This is not the case for the Frank system, with $\mathrm{P}^{\mathrm{CC}}_{\mathrm{f,sys}} = 0.194$ for the canonical system and $\mathrm{P}^{\mathrm{RR}}_{\mathrm{f,sys}} = 0.215$ for the reverse system. Given that the corresponding CMC result is $\mathrm{P}^{\mathrm{CMC}}_{\mathrm{f,sys}} = 0.178$ this leads to the relative impact factor of IF-rel = 11% and the CMC impact factor of IF-CMC = 81%.

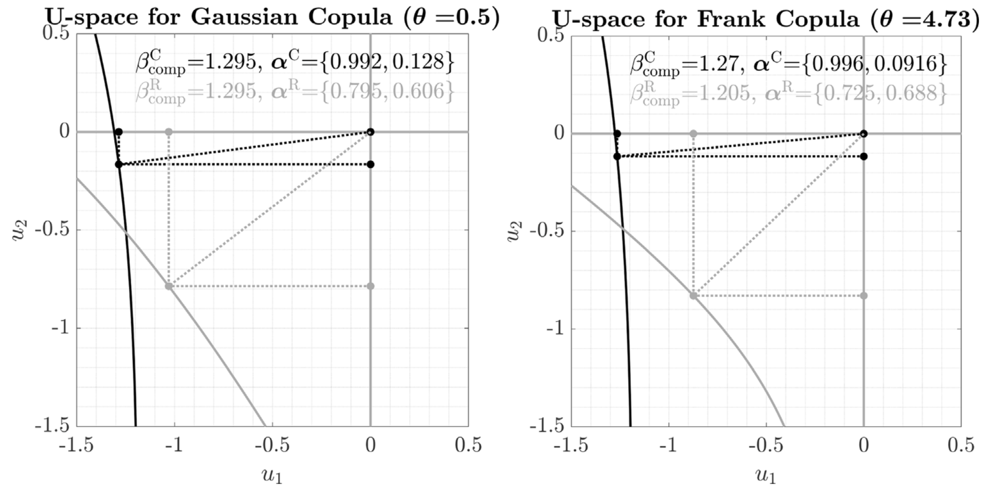

Fig. 1. LSFs in the standard normal **U**-space for the Gaussian (left) and Frank (right) Copula function. Results for the canonical conditioning order are presented in black and results for the reverse conditioning order are presented in grey.

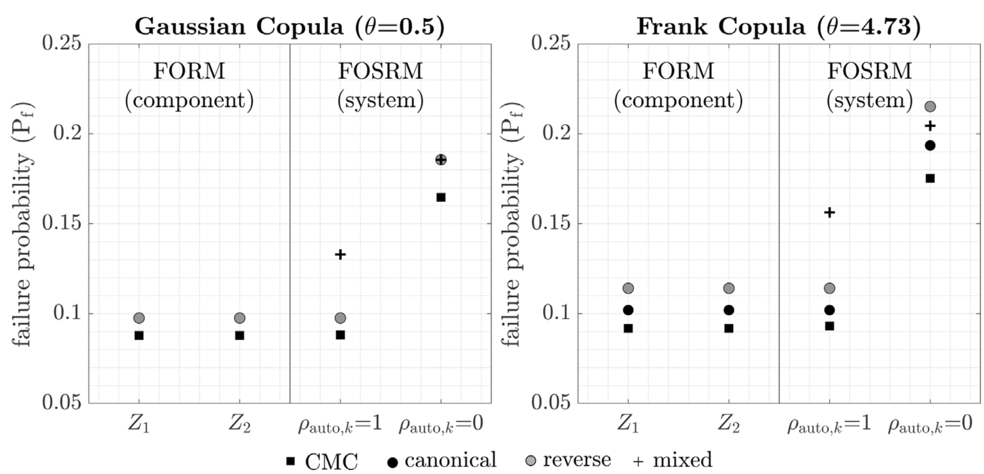

Fig. 2. FOSRM and CMC results for the Gaussian (left) and Frank (right) systems with fully dependent ($\rho_{\text{auto},k} = 1$) and fully independent ($\rho_{\text{auto},k} = 0$) components.

In the case of the Gaussian system, we now see that the mixed systems (RC, CR) lead to the same reliability outcomes as the uniform systems. This is not the case for the Frank system, where the mixed systems essentially represent the average of the canonical (CC) and reverse (RR) systems.

4. Example 2: Parametric study for Frank Copula

4.1. System configuration

In this example we further investigate the impact of the Rosenblatt conditioning order for systems including non-Gaussian Copula functions. We do this on the basis of a parametric study of various system configurations. All investigated systems represent a series system with m identical components and the following LSFs:

$$
\begin{aligned}
Z_1 &= g_1(\mathrm{X}) = c\,R^{(1)} - h^{(1)} - h^{(1)} H^{(1)} \\
&\vdots \\
Z_i &= g_i(\mathrm{X}) = c\,R^{(i)} - h^{(i)} - h^{(i)} H^{(i)} \qquad (16) \\
&\vdots \\
Z_m &= g_m(\mathrm{X}) = c\,R^{(m)} - h^{(m)} - h^{(m)} H^{(m)}
\end{aligned}
$$

where $\{R^{(i)}\}$ represent the resistance variables, and the $\{h^{(i)}\}$ and $\{H^{(i)}\}$ the loading variables. The parameter c is a deterministic optimization constant which is used to achieve a given component reliability level (as obtained via CMC calculations).

Marginal distributions. All resistance variables $\{R^{(i)}\}$ are identically distributed according to a Gaussian distribution with mean value $\mu_R = 1.0$ and standard deviation $\sigma_R = 0.15$. All loading variables $\{h^{(i)}\}$ are identically distributed according to the Gumbel distribution for maxima with mean value $\mu_h = 1.0$ and standard deviation $\sigma_h = 0.2$. All loading parameters $\{H^{(i)}\}$ are modelled with a log-normal distribution with mean value $\mu_H = 1.0$ and standard deviation $\sigma_H = 0.2$. The first requirement (i) of Roscoe's approximation is hereby fulfilled (see Section 2.1.2).

Within-component dependency. Within each component Z_i, the resistance and loading parameters are modelled statistically independent from each other, i.e, no dependency between the variables $\{R^{(i)}, h^{(i)}\}$ or $\{R^{(i)}, H^{(i)}\}$ exists. The loading parameters $h^{(i)}$ and $H^{(i)}$ are modelled statistically dependent according to the Frank Copula with parameter $\theta = -5$.

Auto-correlations. The statistical dependencies between identical variables from different components are described through the autocorrelations $\rho^{(ij)}_{\text{auto},R}$, $\rho^{(ij)}_{\text{auto},h}$ and $\rho^{(ij)}_{\text{auto},H}$ (see Section 2.1.2). For the values of the autocorrelations four system types are assessed:

(1) *The independent system*, where all autocorrelations are taken $\rho^{(ij)}_{\text{auto},R} = \rho^{(ij)}_{\text{auto},h} = \rho^{(ij)}_{\text{auto},H} = 0$ for all $i \neq j$. Although this hypothetical situation is rarely the case in real-life applications, it is often used as a valuable upper-bound estimate.

(2) *The resistance dependent system*, where the autocorrelation between the resistance variables is taken $\rho^{(jj)}_{\text{auto},R} = 1$ and the autocorrelation between the loading variables is taken $\rho^{(ij)}_{\text{auto},h} = \rho^{(ij)}_{\text{auto},H} = 0$. This system could for example be represented by a single structural element loaded by a time-dependent load with renewal time t and with constant resistance over time. The probability that the element fails in T years can then be seen as a system with $m = T/t$ identical components, with fully dependent resistance and fully independent loads.

(3) *The loading dependent system*, where the autocorrelations of the loading variables are equal to $\rho^{(ij)}_{\text{auto},h} = \rho^{(ij)}_{\text{auto},H} = 1$ and the autocorrelation of the resistance-variable is equal to $\rho^{(ij)}_{\text{auto},R} = 0$. This system type could for example be represented by a situation where the structure consists of m identical structural elements all loaded by the same load.

(4) *The fully dependent system*, with $\rho^{(ij)}_{\text{auto},R} = \rho^{(ij)}_{\text{auto},h} = \rho^{(ij)}_{\text{auto},H} = 1$ for all $j \neq k$. Although this system type is rarely the case in reality, its results are often used as a valuable lower-bound estimate of the failure probability.

Cross dependencies. All cross-dependencies are assumed equal to zero, meaning that there exists no statistical dependency between pairs of $\{R^{(i)}, h^{(j)}\}$, $\{R^{(i)}, H^{(j)}\}$ or $\{h^{(i)}, H^{(j)}\}$ for all $j \neq k$. Note however that this can only be totally true in the case of the independent system, and that this assumption is not entirely correct for the other three system types — hereby violating Roscoe's third requirement (see Section 2.1.2). The impact of this violation on the obtained reliability levels is not the topic of this study and therefore not further discussed.

Number of components and component failure probabilities. For each system type, we vary in the number of system components m (ranging from 1 and 10) and component 'target' failure probabilities $\text{P}_{\text{f,comp,t}}$ (ranging from 10^{-5} and 10^{-2}). The component target failure probability here refers to the pre-defined, fixed failure probability of each component, when calculated using CMC. They are achieved by iteratively adjusting the optimization constant c until the desired failure probability is met. It is important to note that the term 'target' is used slightly different here compared to its typical usage in codification. The chosen ranges are assumed to be typical for structural elements designed according to the Eurocode EN1990 [43].

4.2. FORM component approximations (FOSRM-step 1)

Fig. 3 shows the FORM component reliability indices β_{comp} and sensitivity factors $\boldsymbol{\alpha} = \{\alpha_R, \alpha_H, \alpha_h\}$ for the canonical (black dots) and reverse (grey dots) conditioning orders respectively. For the component target reliability $\text{P}_{\text{f,comp,t}} = 10^{-5}$ the canonical conditioning order leads to a higher reliability index than the reverse conditioning order, although the differences appear small. For all other cases the reverse conditioning order leads to slightly higher reliability indices, although again the difference is small. Concerning the sensitivity factors, the canonical conditioning order leads to systematically higher values for α_h than the reverse conditioning order, whereas the opposite holds for α_H. For α_R, however, sometimes the canonical conditioning order leads to higher values, and sometimes the reverse conditioning order.

4.3. Between-component dependency (FOSRM-step 2)

Fig. 4 shows the resulting pairwise correlation coefficients for the canonical pairs (black dots), the reverse pairs (grey dots), and mixed pairs (black crosses). For the *independent systems*, the between-component correlations are adequately represented by $\rho^{(ij)} = 0$ for all $i \neq j$ for both the canonical, reverse, and mixed pairs, as we had previously seen in example 1 as well. For the *resistance dependent systems* and for the cases $\text{P}_{\text{f,comp,t}} = \{10^{-5}, 10^{-4}\}$, canonical pairs lead to higher correlation coefficients than mixed pairs, whereas the opposite holds for the cases $\text{P}_{\text{f,comp,t}} = \{10^{-3}, 10^{-2}\}$. Hereby the correlation coefficients for the mixed pairs lie systematically in between the canonical and reverse pairs for all cases. For the *loading dependent system* and for $\text{P}_{\text{f,comp,t}} = \{10^{-5}, 10^{-4}\}$, reverse pairs lead to higher correlation coefficients than canonical pairs. For the cases $P_{\text{f,comp,t}} = \{10^{-3}, 10^{-2}\}$ the opposite holds, although the differences appear small. The correlation coefficients for the mixed pairs lie systematically below both canonical and reverse pairs. For the *fully dependent system*, both the canonical and reverse pairs adequately result in $\rho^{(ij)} = 1$ for all $i \neq j$, whereas the mixed pairs all result in (sometimes significantly) too low estimates $\rho^{(ij)} < 1$. This was previously addressed in example 1 as well.

4.4. System failure probabilities and impact factors (FOSRM-step 3)

Fig. 5 shows the failure probabilities for all investigated system types, component target probabilities, and system sizes. Fig. 6 shows the corresponding absolute, relative, and CMC impact factors, with each line type specifying a different component target probability. For Fig. 5 we note that for the component target reliability $\text{P}_{\text{f,comp,t}} = 10^{-5}$ the FOSRM several results are omitted (mixed systems only), as the evaluation of Eq. (5) yielded too low accuracy. In the following section we will carefully discuss the results in the light of the research questions.

5. Discussion of results

In this section, we examine the results of examples 1 and 2 in the context of the research questions outlined in Section 1. Initially, we provide an objective evaluation of the results, followed by interpretations and conclusions relevant to engineering practice in Section 6.

Does the choice of the conditioning order only affect systems with non-Gaussian Copula functions, or also those with (solely) Gaussian Copula functions?

Example 1 showed that the choice of the conditioning order influences both systems with Gaussian and non-Gaussian Copula functions (in casus: Frank). For the Gaussian Copula function, the FOSRM failure probabilities were not affected as long as uniform systems were employed (i.e., identical conditioning order in each of the individual components). For the Frank Copula function, however, the FOSRM failure probabilities differed systematically for the different conditioning orders, even though uniform systems were employed. Notably, for both the Gaussian and Frank Copula function, the application of mixed systems appeared highly vulnerable to the conditioning order, in some cases leading to significant differences compared to uniform systems. This is further addressed in the last research question.

What order of magnitude is the difference in FOSRM results when one conditioning order is applied as compared to the other?

The impact of the conditioning order on FOSRM outcomes was assessed using absolute (IF-abs), relative (IF-rel), and CMC (IF-CMC) impact factors. For both examples 1 and 2, the impact factors appeared strongly dependent on the chosen system configuration. Focusing on example 2, we find *absolute impact factors* between roughly IF-abs ≈ 0 for up to IF-abs ≈ 20. This indicates that while differences at the system level may be negligible in some cases, they can be significantly magnified in others.

Also the *relative impact factors* varied significantly over the investigated system types, with values close to IF-rel ≈ 0 up to IF-rel $>$ 30%. For the vast majority of cases, relative impact factors were found between IF-rel $\approx 10\%$ and 30%, suggesting that one can expect at least a 10% difference in reliability outcomes if the one conditioning order was applied instead of the other. These values are higher than those reported in previous literature, such as in [27] where IF-rel would be as high as 12%.

We find similar results for the *CMC impact factors*, with IF-CMC ≈ 0 for some cases, up to IF-rel $>$ 30% for other cases — again with the

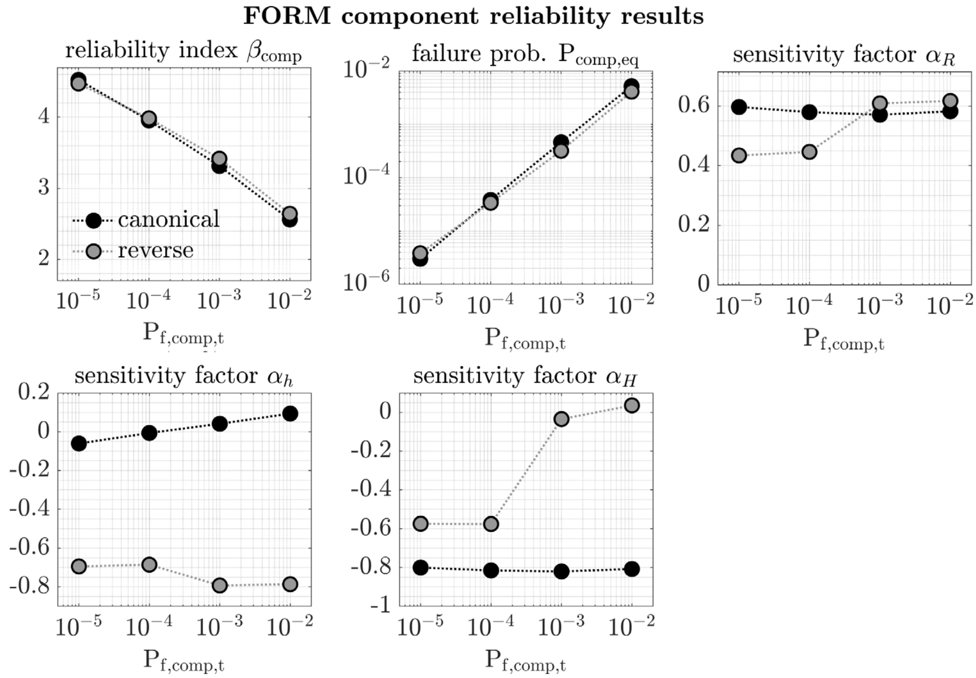

Fig. 3. FORM component results for different target failure probabilities. The obtained FORM results form the basis of the FOSRM calculations in the following sections.

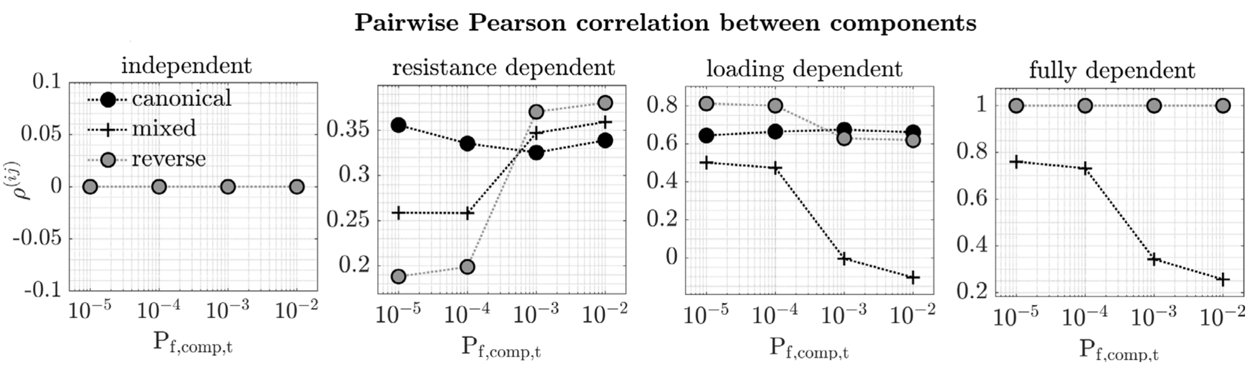

Fig. 4. Pairwise Pearson correlation between components.

majority of investigated cases between 10% and 30%. For example 1, however, we find much larger values, with CMC impact factors of over IF-CMC > 80%. This shows that for some cases, the differences in conditioning order might, in fact, have the same order of magnitude as the general simplifications inherent to FOSRM.

Does the impact of the conditioning order systematically increase with an increasing number of system components?

In example 2 we investigated the effect of different conditioning orders for increasing system size. Focusing on Fig. 6, we observe that in most cases, an increasing number of system components leads to a rise in *absolute impact factors*, particularly for the independent or resistance dependent systems. However, this systematic increase is not observed for all systems, such as for the dependent systems and some loading dependent systems. For the dependent systems, this can easily be explained by the fact that the pairwise correlation was adequately determined as $\rho^{(ij)} = 1$, naturally leading to identical component and system failure probabilities for all system sizes (see Fig. 4). More notable is the pattern observed for the loading dependent system and $P_{f,comp,t} = 10^{-5}$, which shows an initial slight increase in IF-abs up to $m = 4$, followed by a decrease down to approximately zero at $m = 9$, and then again increase for $m = 10$. To understand this behaviour, we refer to Fig. 5, which shows that for $m < 9$ the reverse systems lead to higher failure probabilities, but for $m > 9$ the canonical systems lead to higher failure probabilities. This 'flip' can be explained by the fact that on the component level, the reverse conditioning order leads to the highest failure probability, but also the highest correlation coefficient (see Figs. 3 and 4 respectively). Thus, the reverse system behaves more like a dependent system than the canonical system, resulting in a slower increase in system failure probability with increasing system size.

In contrast to the absolute impact factors, the *relative and CMC impact factors* appear less sensitive to the increasing system size. For the independent, resistance dependent, and dependent systems, IF-abs and IF-CMC remain approximately constant with increasing system size. For the loading dependent system, however, both a decrease in relative impact with increasing system size is observed, e.g. for $P_{f,comp,t} = \{10^{-2}, 10^{-3}\}$, as well as an increase, e.g. for $P_{f,comp,t} = \{10^{-4}\}$. This latter observation suggests that adding more components to the system would likely result in higher relative and CMC impact factors, potentially exceeding 30%.

What happens when different system components are assessed with different conditioning orders (i.e. mixed systems)?

The First Order System Reliability Method (FOSRM) implicitly assumes that each component is assessed using the same iso-probabilistic transformation method, making it logical to apply a consistent Rosenblatt conditioning order across all components before aggregating them into the FOSRM system reliability. This uniform approach is typically

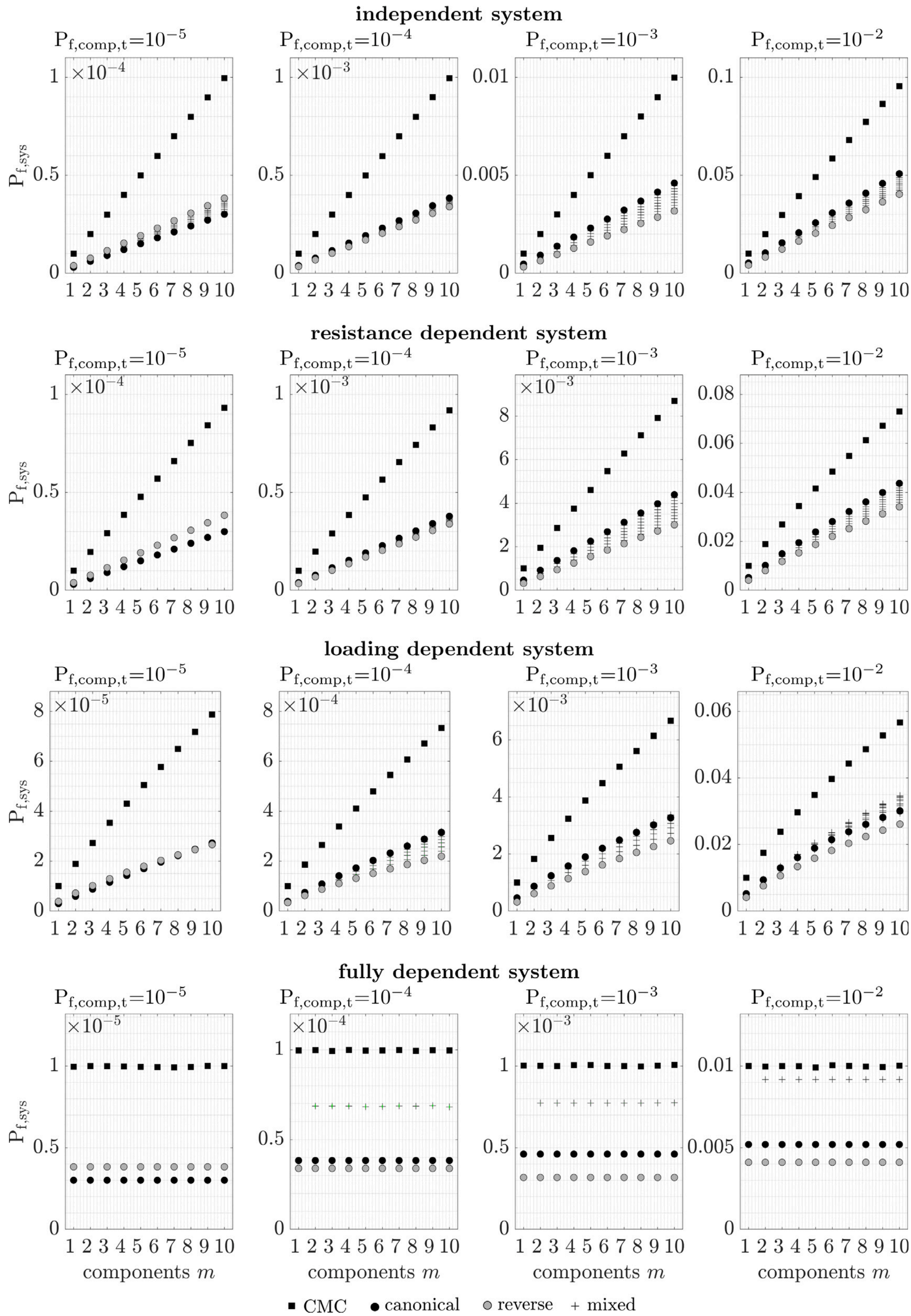

Fig. 5. FOSRM and CMC results for the investigated systems in example 2.

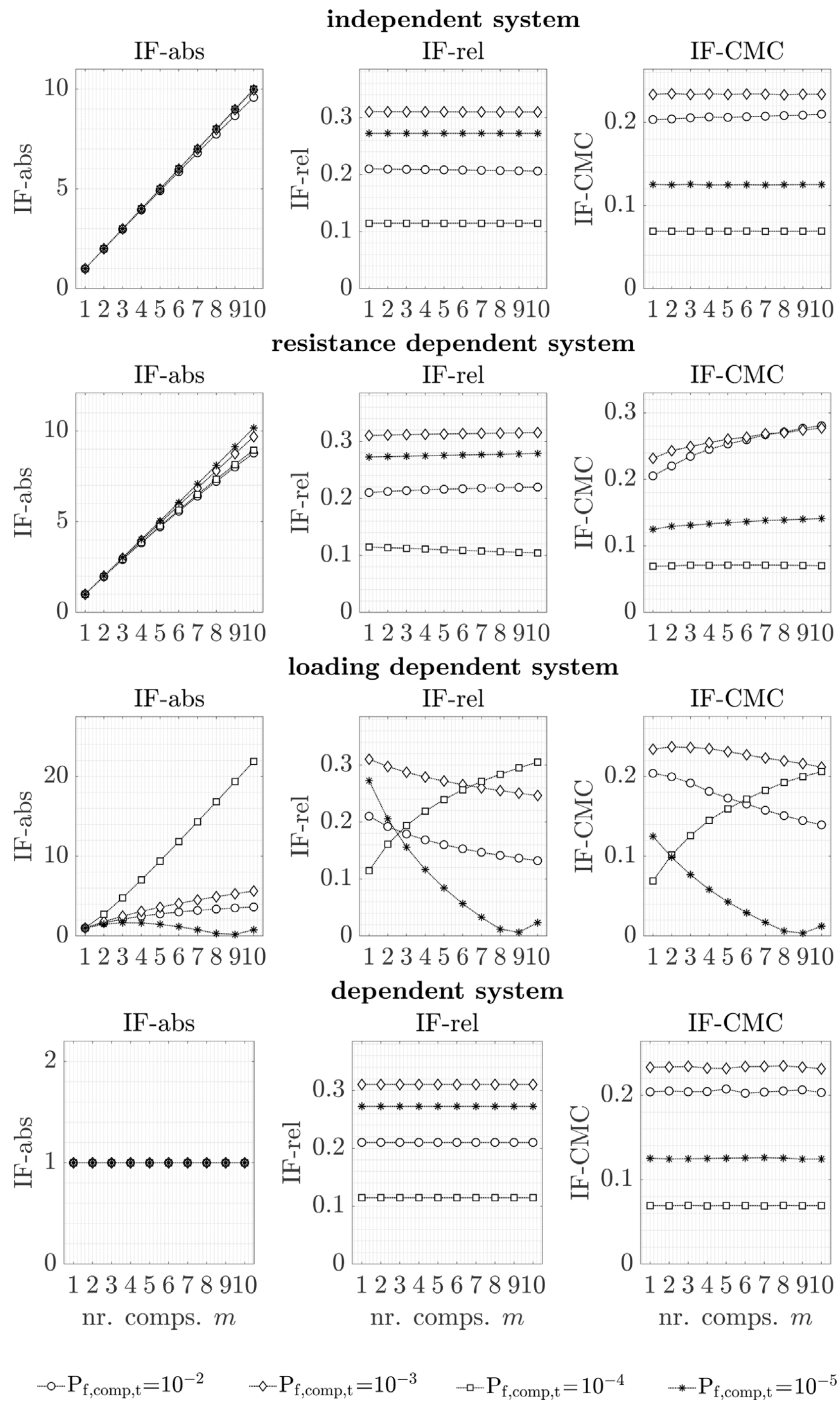

Fig. 6. Absolute (IF-abs), relative (IF-rel), and CMC (IF-CMC) impact factors for the investigated systems in example 2.

followed in engineering practice, especially for series systems with identical components like those in our study. However, for complex systems where different experts assess different components, mixed systems are more likely to occur.

In both examples 1 and 2, we observe that the degree of discrepancy between mixed and uniform systems strongly depends on the investigated system type. For the independent systems, the results of the mixed systems were found to lie systematically in between the results of the uniform (canonical and reverse) systems. Also for the resistance and loading dependent systems, the results of the mixed systems were found to lie systematically in between, or slightly above the results of the uniform systems. For dependent systems, however, we notice clear discrepancies between the uniform and mixed systems; the mixed systems systematically yielded a lower between-component

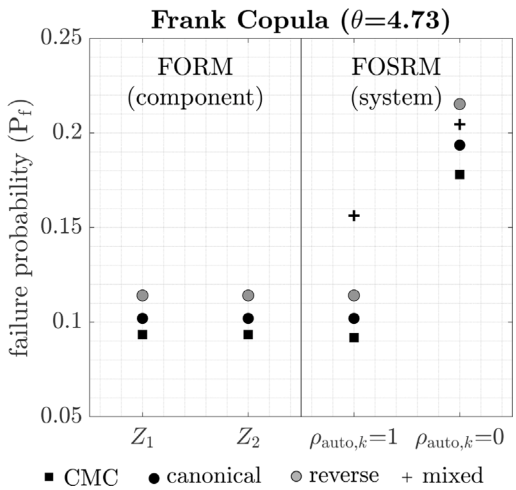

Fig. 7. Illustration of how the results (and main observations) from example 1 change, when the second component LSF is changed into $Z_2 = X_2 - 0.5X_1$.

correlation and a higher FOSRM system failure probability than the uniform systems. This was found to be true for both the Frank and Gaussian Copula functions.

It is important to note that the observations for the (identical) series systems here may not necessarily generalize to other system cases. Suppose for example we would change the second LSF from example 1 to $Z_2 = X_2 - 0.5X_1$, we would, in fact, obtain that for the Frank Copula the mixed systems lie *in between* the uniform systems for the dependent case ($\rho_{\text{auto},k} = 1$), and *outside* the uniform systems for the independent case ($\rho_{\text{auto},k} = 0$), see Fig. 7. This is the opposite of what we previously found in example 1.

6. Conclusions and recommendations

6.1. Conclusions

This study investigated the impact of the Rosenblatt conditioning order on FOSRM system reliability calculations. This was done based on two academic examples, including series systems with different dependency structures (Gaussian, Frank), component reliabilities, and system sizes. We aimed to mirror realistic case studies, incorporating non-linear LSFs and typical distribution functions (Exponential, Gumbel, Normal, log-Normal), to ensure practical relevance in the assessed orders of magnitude. Although the obtained results are specific to the investigated cases, they provide valuable insights to engineering practice.

For the systems including the Frank Copula function, we found that different conditioning orders systematically yielded different FOSRM results. The observed differences were greater than previously reported, with most cases showing differences between 10% and 30% in the estimated failure probability. Notably, for some systems, the differences systematically increased with system size, indicating that larger systems might exhibit greater variations than reported here.

When comparing the differences to other simplifications inherent to FOSRM, such as the linearization of component LSFs and the assumption of the Pearson correlation matrix, we found a significant case-dependency. For the typical engineering examples with non-linear LSFs, the obtained differences ranged between 10% and 30%, indicating that the effect of the conditioning order was subordinate to the other simplifications inherent to FOSRM. For the linear example, however, the observed differences were as high as 81%, indicating that, in some cases, the differences between conditioning orders might be as significant as the general simplifications inherent to FOSRM.

Furthermore, we qualitatively analysed the effects of assessing different system components with varying conditioning orders (mixed systems). For the typical engineering examples, the observed differences between the uniform and mixed systems appeared small for the majority of cases. For some situations, however, the application of mixed systems led to significant differences in the estimated failure probability as compared to the uniform systems. This finding was consistent for both the Frank Copula function and the Gaussian Copula function.

6.2. Recommendations

Prior to this study, the impact of the Rosenblatt conditioning order on FOSRM system reliability calculations remained unclear. Consequently, the selection of the conditioning order was often overlooked or inadequately considered in engineering practice. The results from this study indicate that neglecting the conditioning order may result in notable difference in FOSRM outcomes. The extent to which the obtained differences are relevant to engineering practice, however, strongly depends on the desired level of accuracy. While deviations of up to 10% or 20% are typically deemed acceptable in engineering, discrepancies exceeding 30% may warrant closer scrutiny — especially when safety or economic considerations are at stake.

Regardless of the required accuracy, we recommend adopting best practices to ensure consistent and transparent safety assessments, thereby preventing ambiguity in FOSRM reliability results. To achieve this, it is important to establish predetermined agreements and documentation on the chosen conditioning order. For projects requiring high accuracy, a systematic exploration of potential conditioning orders is advised to facilitate informed decision-making. This approach becomes particularly relevant when dealing with larger systems with many dependent random variables and when different components are assessed by various experts.

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CRediT authorship contribution statement

N.E. Meinen: Methodology, Investigation, Formal analysis, Conceptualization, Visualization, Validation, Software, Resources, Writing – original draft, Writing – review & editing. **R.D.J.M. Steenbergen:** Writing – review & editing, Supervision, Resources, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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