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Minimizing the effective graph resistance by adding links is NP-hard

Robert E. Kooij [a,b,*], Massimo A. Achterberg [a]

[a] Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA, Delft, the Netherlands
[b] Unit ICT, Strategy & Policy, TNO (Netherlands Organisation for Applied Scientific Research), P.O. Box 96800, 2509 JE, The Hague, the Netherlands

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The effective graph resistance, also known as the Kirchhoff index, is metric that is used to quantify the robustness of a network. We show that the optimisation problem of minimizing the effective graph resistance of a graph by adding a fixed number of links, is NP-hard.

1. Introduction

Many network metrics have been utilised to quantify the robustness of a network, see for instance [1], [2], [11], [19], [20]. Freitas et al. [6] classify robustness metrics into three types: metrics based on structural properties, such as edge connectivity or diameter; metrics based on the spectrum of the adjacency matrix, such as the spectral radius or spectral gap; and metrics based on the spectrum of the Laplacian matrix, for instance the algebraic connectivity and the effective graph resistance. In this paper we consider the following optimisation problem: how to augment a given graph G by adding at most k links, such that the robustness metric of the augmented network is optimal. As robustness metric we consider the effective graph resistance R_G, also known as the Kirchhoff index, see Ellens et al. [4]. The effective graph resistance not only covers the shortest path between any pair of nodes, but incorporates all paths between any two nodes. Because in addition R_G decreases upon the addition of a link to the graph [9], this makes the effective graph resistance a good metric to evaluate the robustness of a network.

Predari et al. refer to the optimisation problem at hand as k-Graph Robustness Improvement Problem (k-GRIP) [18], in which one has to decide where k links are to be added to a given network G, such that the robustness metric is optimised. Several researchers investigated k-GRIP for specific robustness metrics. For instance, Wang et al. [21] considered 1-GRIP, with as robustness metric the second-smallest eigenvalue of the Laplacian matrix, which was coined the algebraic connectivity by Fiedler [5]. They suggest several strategies to decide which single link to add to the network, in order to increase the algebraic connectivity as much as possible. A nice overview of k-GRIP for the algebraic connectivity is presented in [12]. The NP-hardness of k-GRIP for the algebraic connectivity was proved in [14].

For the effective graph resistance, 1-GRIP was considered by Wang et al. [22]. They investigated different strategies, based upon topological and spectral properties of the graph, to determine the most optimal link to add, and derived a lower bound for R_G after adding a single link. Pizzuti et al. [16], [17] proposed and evaluated several genetic algorithms to find the most optimal link to add, in order to minimize R_G. Clemente et al. [3] studied k-GRIP for the effective graph resistance and gave lower bounds for R_G upon the addition of k links, under some mild conditions for k. For $k = 1$ the lower bound in [3] clearly outperforms the lower bound in [22]. Predari et al. [18] also consider k-GRIP for the effective graph resistance. They focus on heuristics for k-GRIP, based upon sampling and a fast approximation method, to compute the effective graph resistance.

Although for some choices of the robustness metric, k-GRIP is known to be NP-hard, to the best of our knowledge this has not been proved yet for the effective graph resistance. The aim of this paper is to prove that augmenting a given graph G by adding k links, in order to minimize the effective graph resistance, is NP-hard. Note that [9] considered the optimisation problem of the effective graph resistance in the case of weighted links. They pro-

* Corresponding author at: Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA, Delft, the Netherlands.

E-mail addresses: R.E.Kooij@tudelft.nl (R.E. Kooij), M.A.Achterberg@tudelft.nl (M.A. Achterberg).

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vide an efficient (polynomial-time) algorithm under the condition that the sum of the weights is constant. In this paper, however, the graph G is considered unweighted and simple.

2. Definitions and main result

In this paper we consider undirected, connected simple graphs $G = (V, E)$ without self-loops. Here V denotes the set of N nodes, while E is the set of L links connecting node pairs of V. The notation $i \sim j$ indicates that nodes i and j are adjacent in G. We let $G^c = (V, E^c)$ denote the complementary graph of G, where $E^c = \{(u, v)|u, v \in V, u \neq v, (u, v) \notin E\}$. The adjacency matrix A of G is an $N \times N$ symmetric matrix with elements a_{ij} that are either 1 or 0 depending on whether there is a link between nodes i and j or not. The Laplacian matrix Q of G is an $N \times N$ symmetric matrix $Q = \Delta - A$, where $\Delta = diag(d_i)$ is the $N \times N$ diagonal degree matrix with the elements $d_i = \sum_{j=1}^{N} a_{ij}$. The eigenvalues of Q are all real and non-negative and can be ordered as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

Interpreting the graph G as an electrical network whose links are resistors of 1Ω, the effective resistance ω_{ij} between node i and j can be computed based on Kirchoff's circuit laws. Then the *effective graph resistance* R_G, also known as the *Kirchhoff index*, is defined as the sum of the resistances over all node pairs [10]

$$R_G(G) = \sum_{1 \leq i < j \leq N} \omega_{ij}. \tag{1}$$

Klein and Randić [10] showed that the effective graph resistance can also be computed using the Laplacian eigenvalues λ_k of the graph G as

$$R_G(G) = N \sum_{k=2}^{N} \frac{1}{\lambda_k}. \tag{2}$$

Ellens et al. [4] argued that the effective graph resistance is an appropriate robustness metric. Note that the smaller the value of R_G the larger the robustness of the network. The smallest value of the effective graph resistance for a graph on N nodes is obtained for the complete graph K_N and satisfies $R_G(K_N) = N - 1$. We will show in this paper that adding a specified number of links to a given graph, in order to minimize the effective graph resistance, is NP-hard. We will now give an explicit description of the considered optimisation problem.

Problem 1 *(Minimum effective graph resistance augmentation problem).* Given an undirected, connected, simple graph $G = (V, E)$, a non-negative integer k and a non-negative threshold t, is there a subset $B \subseteq E^c$ of size $|B| \leq k$ such that the graph $H = (V, E \cup B)$ satisfies $R_G(H) \leq t$?

Problem 1 is clearly in NP, because given a graph G and the set of added links B, the correctness of the given solution can be verified by computing the eigenvalues of the Laplacian matrix, which is an $\mathcal{O}(N^3)$ operation. Then simply computing (2) and comparing the outcome with the given threshold t verifies the solution. Thus the minimum effective graph resistance augmentation problem is in NP.

Problem 1 is the decision version of the following optimisation problem: Given an undirected, connected, simple graph $G = (V, E)$ and a non-negative threshold t, find a set of currently non-existent links of minimum size to add to G such that the effective graph resistance R_G of the augmented graph is at most t. We prove in this work that Problem 1 is NP-hard, which immediately implies that the corresponding optimisation problem is also NP-hard. Thus, the problem of adding a specified number of links to a graph to minimize the effective graph resistance is also NP-hard. We now state the main result of the paper.

Theorem 2. *The minimum effective graph resistance augmentation problem is NP-hard.*

3. Proof of Theorem 2

The proof of Theorem 2 heavily relies on the proof of the NP-hardness of the maximum algebraic connectivity augmentation problem, as given in [14]. The proof is by reduction of our augmentation problem to a problem for which NP-hardness has been proved, namely the 3-colorability problem, see [7]. For our proof we will use a construction and a lemma from [14] and two additional lemmas.

Construction. *[14] Given a graph $G = (V, E)$ with $n > 1$ nodes and m links, a graph $G' = (V', E')$ is constructed which consists of three disjoint copies G_0, G_1 and G_2 of G. This implies that each node $v \in V$ is copied to a node $v_i \in G_i$ and each link $(u, v) \in E$ is copied to $(u_i, v_i) \in G_i$, for $i = 0, 1, 2$. By construction the graph G' has $3n$ nodes and $3m$ links. We now consider the minimum effective graph resistance augmentation problem on G' with $k = 3n^2 - 3m$, such that the augmented graph H has at most $3n^2$ links and $t = \frac{9n-5}{2}$.*

Now, let $K_{n,n,n}$ denote the complete tripartite graph. In order to prove that the minimum effective graph resistance augmentation problem can be reduced to the 3-colorability problem, we will use the following three lemmas.

Lemma 3. *[14] There exists a subset $B \subseteq (E')^c$ of size $|B| \leq k$ such that $H = (V', E' \cup B)$ is (isomorphic to) $K_{n,n,n}$ if and only if G is 3-colorable.*

Lemma 4. *[13] Let G be a simple connected graph with $N \geq 2$ nodes and L links. Then*

$$R_G(G) \geq \frac{N^2(N-1)}{2L} - 1,$$

with equality if and only if $G \cong K_N$, or $G \cong K_{N/2,N/2}$, or $G \in \Gamma_d$.

Here, Γ_d denotes a special class of d-regular graphs defined in [15]. Let $M(i)$ be the set of all neighbours of the node i, that is, $M(i) = \{k|k \in V, k \sim i\}$, where V denotes the set of nodes of the graph. Then for every $1 \leq d \leq n-1$ the set Γ_d denotes the set of all d-regular graphs with diameter 2 and satisfying $|M(i) \cap M(j)| = d$ for every pair of nodes i, j that are not adjacent, i.e. $i \nsim j$.

Lemma 5. *The complete tripartite graph $K_{n,n,n}$ on $3n$ nodes has effective graph resistance $R_G(K_{n,n,n}) = \frac{9n-5}{2}$.*

Proof. We compute the effective graph resistance R_G of the complete tripartite graph $K_{n,n,n}$ using Eq. (1). Gervacio [8] derived the effective resistance between nodes in complete multipartite graphs as:

$$\omega_{ij} = \frac{2}{N - m_i}, \qquad \text{if } i, j \text{ are in the same partition}$$

$$\omega_{ij} = \frac{(N-1)(2N - m_i - m_j)}{N(N - m_i)(N - m_j)}, \qquad \text{otherwise}$$

where m_i and m_j represent the size of the partition of node i and j respectively. In our case, $N = 3n$ and $m_i = m_j = n$. The number of node pairs in the same partition equals $3n(n-1)/2$ and the number of pairs outside of the same partition equals $3n^2$. Then the effective graph resistance of the complete tripartite graph exactly equals $R_G(K_{n,n,n}) = \frac{9n-5}{2}$. □

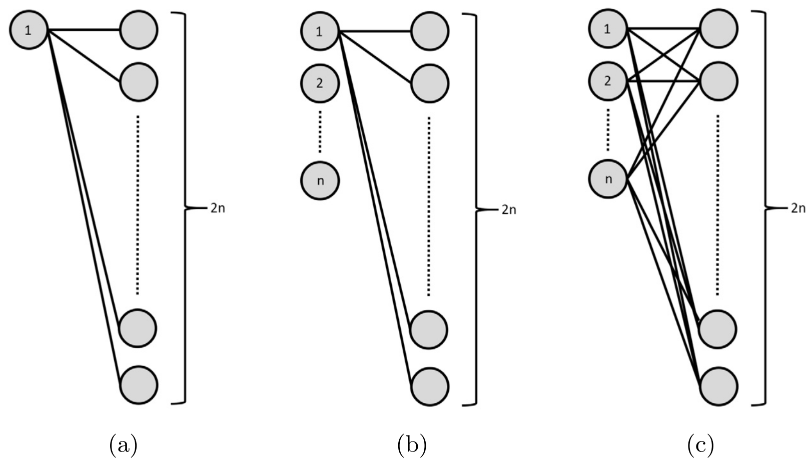

Fig. 1. (a) Node 1 and its $2n$ neighbours. (b) Node 1, its $2n$ neighbours and the $n-1$ remaining nodes. (c) Nodes $\{1, \cdots, n\}$ and their connections to the other $2n$ nodes.

Lemma 6. *A graph $H = (V, E)$ with $N = 3n$ nodes and $L \leq 3n^2$ links for $n > 1$ satisfies $R_G(H) \leq \frac{9n-5}{2}$ if and only if H is (isomorphic to) $K_{n,n,n}$.*

Proof. The backward direction is satisfied by Lemma 5.

To prove the forward direction, using $N = 3n$, $L \leq 3n^2$ and Lemma 4, it follows $R_G(H) \geq \frac{9n^2(3n-1)}{6n^2} - 1 = \frac{9n-5}{2}$. By the condition $R_G(H) \leq \frac{9n-5}{2}$, we deduce that $R_G(H) = \frac{9n-5}{2}$. Also it follows that $L = 3n^2$ because $N = 3n$ and $L < 3n^2$ would imply $R_G(H) > \frac{9n-5}{2}$ according to Lemma 4. Therefore the average degree of H equals $2n$. Since $R_G(H)$ is equal to the lower bound given in Lemma 4, H is either the complete graph K_{3n}, the complete bipartite graph $K_{3n/2,3n/2}$ or it is a $2n$-regular graph belonging to the class Γ_{2n}. First, assume $H \cong K_{3n}$. The number of links of K_{3n} equals $\frac{3n(3n-1)}{2}$ which, for $n > 1$, is larger than $3n^2$, the number of links of H. Therefore $H \ncong K_{3n}$. Next assume $H \cong K_{3n/2,3n/2}$, which can only hold for n even. Then the number of links of $K_{3n/2,3n/2}$ equals $\frac{9n^2}{4}$ which is always smaller than $3n^2$, the number of links of H. Therefore $H \ncong K_{3n/2,3n/2}$. Hence we conclude that the graph H is $2n$-regular and belongs to the class Γ_{2n}.

We will now show that H is isomorphic to $K_{n,n,n}$. We start with an arbitrary node of H and label it as node 1. Because H is $2n$-regular, node 1 has exactly $2n$ neighbours, see Fig. 1a.

The remaining $n-1$ nodes, other than node 1 and its $2n$ neighbours, cannot be adjacent to node 1 because it already has degree $2n$, by construction. We now label these nodes as nodes 2 until n, see Fig. 1b. Now, because H belongs to the class Γ_{2n} and nodes 2 until n are not adjacent to node 1, each of the nodes 2 until n has exactly the same neighbours as node 1, see Fig. 1c.

Next, take an arbitrary node outside the set $\{1, 2, \cdots, n\}$ and label it as $n+1$. To obtain degree $2n$, node $n+1$ needs to be adjacent to n nodes outside the nodes $\{1, 2, \cdots, n\}$. We label this set of n adjacent nodes as $\{2n+1, \cdots, 3n\}$, see Fig. 2a.

Finally, every node not in $\{1, 2, \cdots, n+1\} \cup \{2n+1, \cdots, 3n\}$ needs to share with node $n+1$ its neighbours $\{2n+1, \cdots, 3n\}$, see Fig. 2b.

Denote by S_i the nodes labelled as $\{n(i-1)+1, n(i-1)+2, \cdots, n(i-1)+n\}$, for $i = 1, 2, 3$. Then $|S_i| = n$, every node pair within S_i is not adjacent and for every $i \neq j$ all nodes in S_i are adjacent to all nodes in S_j. This proves that H is a complete tripartite graph $K_{n,n,n}$. $\square$

Finally, Theorem 2 follows from combining Lemma 3 and 6.

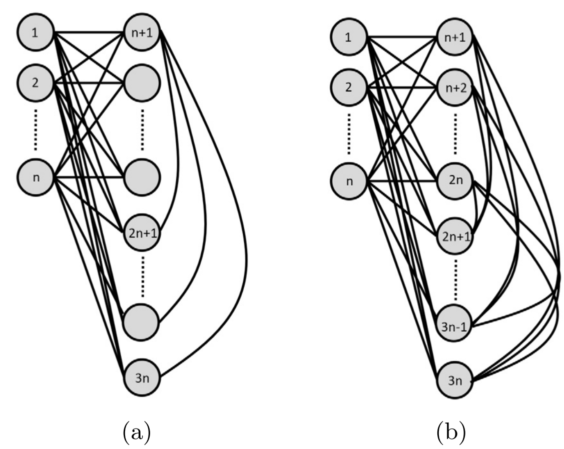

Fig. 2. (a) Nodes $\{1, \cdots, n\}$, their connections to the other $2n$ nodes and the additional n connections of node $n+1$. (b) All connections in graph H.

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The authors have no relevant financial or non-financial interests to disclose.

Data availability

No data was used for the research described in the article.

References

[1] O. Cats, G.-J. Koppenol, M. Warnier, Robustness assessment of link capacity reduction for complex networks: application for public transport systems, Reliab. Eng. Syst. Saf. 167 (2017) 544–553.

[2] H. Cetinay, C. Mas-Machuca, J.L. Marzo, R. Kooij, P. Van Mieghem, Comparing Destructive Strategies for Attacking Networks, Springer International Publishing, 2020, pp. 117–140.

[3] G.P. Clemente, A. Cornaro, Bounding robustness in complex networks under topological changes through majorization techniques, Eur. Phys. J. B 93 (114) (2020) 1–12.

[4] W. Ellens, F.M. Spieksma, P. Van Mieghem, A. Jamakovic, R.E. Kooij, Effective graph resistance, Linear Algebra Appl. 435 (10) (2011) 2491–2506, Special Issue in Honor of Dragos Cvetkovic.

[5] M. Fiedler, Algebraic connectivity of graphs, Czechoslov. Math. J. 23 (1973) 298–305.

[6] S. Freitas, D. Yang, S. Kumar, H. Tong, D.H. Chau, Graph vulnerability and robustness: a survey, IEEE Trans. Knowl. Data Eng. (2022), https://doi.org/10.1109/TKDE.2022.3163672.
[7] M. Garey, D. Johnson, L. Stockmeyer, Some simplified NP-complete graph problems, Theor. Comput. Sci. 1 (3) (1976) 237–267.
[8] S.V. Gervacio, Resistance distance in complete n-partite graphs, Discrete Appl. Math. 203 (2016) 53–61.
[9] A. Ghosh, S. Boyd, A. Saberi, Minimizing effective resistance of a graph, SIAM Rev. 50 (1) (2008) 37–66.
[10] D.J. Klein, M. Randić, Resistance distance, J. Math. Chem. 12 (1993) 81–95.
[11] Y. Koç, M. Warnier, P. Van Mieghem, R.E. Kooij, F.M.T. Brazier, A topological investigation of phase transitions of cascading failures in power grids, Phys. A, Stat. Mech. Appl. 415 (2014) 273–284.
[12] G. Li, Z.F. Hao, H. Huang, H. Wei, Maximizing algebraic connectivity via minimum degree and maximum distance, IEEE Access 6 (2018) 41249–41255.
[13] I. Milovanović, E. Milovanović, On some lower bounds of the Kirchhoff index, MATCH Commun. Math. Comput. Chem. 78 (2017) 169–180.
[14] D. Mosk-Aoyama, Maximum algebraic connectivity augmentation is NP-hard, Oper. Res. Lett. 36 (6) (2008) 677–679.
[15] J. Palacios, Some additional bounds for the Kirchhoff index, MATCH Commun. Math. Comput. Chem. 75 (2016) 365–372.
[16] C. Pizzuti, A. Socievole, A genetic algorithm for improving robustness of complex networks, in: 2018 IEEE 30th International Conference on Tools with Artificial Intelligence (ICTAI), 2018, pp. 514–521.
[17] C. Pizzuti, A. Socievole, Incremental computation of effective graph resistance for improving robustness of complex networks: a comparative study, in: H. Cherifi, R.N. Mantegna, L.M. Rocha, C. Cherifi, S. Micciche (Eds.), Complex Networks and Their Applications XI, Springer International Publishing, Cham, 2023, pp. 419–431.
[18] M. Predari, R.E. Kooij, H. Meyerhenke, Faster greedy optimization of resistance-based graph robustness, in: Proceedings of the 2022 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM '22), IEEE Press, 2022, pp. 1–8.
[19] D.F. Rueda, E. Calle, J.L. Marzo, Robustness comparison of 15 real telecommunication networks: structural and centrality measurements, J. Netw. Syst. Manag. 25 (2) (2017) 269–289.
[20] C.M. Schneider, A.A. Moreira, J.S. Andrade, S. Havlin, H.J. Herrmann, Mitigation of malicious attacks on networks, Proc. Natl. Acad. Sci. 108 (10) (2011) 3838–3841.
[21] H. Wang, P. Van Mieghem, Algebraic connectivity optimization via link addition, in: Proceedings of the 3rd International Conference on Bio-Inspired Models of Network, Information and Computing Systems, BIONETICS '08, Brussels, BEL, ICST (Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering), 2008.
[22] X. Wang, E. Pournaras, R.E. Kooij, P. Van Mieghem, Improving robustness of complex networks via the effective graph resistance, Eur. Phys. J. B 87 (221) (2014).