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SHORT COMMUNICATION

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Closed form radial return algorithm for plane stress elastic-plastic analyses

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Abstract
Plane stress numerical simulations of elastic-plastic deformations are commonly based on in-plane projected equations. It is shown that in case of the isotropic Von Mises material model the numerical procedure can be simplified and faster when the full 3D yield criterion is used. Plane stress is preserved by plane stress trial stresses in the return mapping procedure.

KEYWORDS
finite elements, plane stress, plasticity, return mapping

1 | INTRODUCTION

Plane stress numerical simulations of elastic-plastic deformations are commonly based on in-plane projected equations, that is, direct inclusion of the plane stress constraints into the three-dimensional elastic predictor and plastic corrector algorithm. In case of a full three-dimensional formulation (without plane stress constraints) and a Von Mises yield criterion, a closed form return mapping can be obtained, resulting in a linear equation for the scalar plastic multiplier and subsequently an explicit constitutive function[1(pg 223 eq. 7.101 and 7.103)]. An algorithm for plane stress using the full 3D constitutive formulation is introduced by de Borst,[2] by enforcing plane stress at the structural level rather than in the constitutive integration algorithm. In this approach the plane stress constraint is satisfied only at converged equilibrium conditions. For an explicite code this approach is not appropriate as there is no equilibrium iteration. By applying the in-plane projected equations a nonlinear equation for the scalar plastic multiplier is obtained[1(remark 9.8 pg 376)]. In this paper it is shown that yet a linear equation for the plastic multiplier and an explicit constitutive function that satisfies the plane stress constraints can be obtained. Section 2.2 summarizes the plane stress formulation by De Souza Neto.[1] Consequences of plane stress projected equations are presented in Section 2.3, and an explicit closed form radial return algorithm for plane stress in Section 2.4.

2 | PLANE STRESS RETURN MAPPING

2.1 | The Von Mises yield criterion

The Von Mises yield function is commonly defined by

$$\Phi = \sqrt{3J_2} - \sigma_{\mathrm{yld}}\left(\epsilon^p\right). \tag{1}$$

In tensor notation $J_2 = \frac{1}{2}\mathbf{s} : \mathbf{s}$ is the second invariant of the deviator stress:

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}tr(\boldsymbol{\sigma})\mathbf{I}, \tag{2}$$

$\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{I}$ is the second order unit tensor and $\sigma_{\mathrm{yld}}(\epsilon^p)$ is the unidirectional yield stress function, depending on the (equivalent) plastic strain ϵ^p.

2.2 | Recapitulation of plane stress strategy from Reference 1

The yield function projected to plane stress in matrix-vector notation is:

$$\Phi^* = q^* - \sigma_{\mathrm{yld}}\left(\epsilon^p\right) = \sqrt{\frac{3}{2}\boldsymbol{\sigma}^{*T}\mathbf{P}^*\boldsymbol{\sigma}^*} - \sigma_{\mathrm{yld}}\left(\epsilon^p\right). \tag{3}$$

For convenience a squared form of the yield function is used instead:

$$\Psi^* = \frac{1}{2}\boldsymbol{\sigma}^{*T}\mathbf{P}^*\boldsymbol{\sigma}^* - \frac{1}{3}\sigma_{\mathrm{yld}}^2\left(\epsilon^p\right) \tag{4}$$

within matrix form

$$\mathbf{P}^* = \frac{1}{3}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{5}$$

and the Cauchy stress vector

$$\boldsymbol{\sigma}^{*T} = [\sigma_x, \sigma_y, \tau_{xy}]. \tag{6}$$

In order to distinguish between plane stress and full 3D an asterisk (*) is added to plane stress related symbols. Besides Φ for the root form, Ψ is denoted for the square form respectively, (3) and (4). The in-plane plastic strain increment is then:

$$\Delta\boldsymbol{\varepsilon}^{*p} = \Delta\lambda\frac{\partial\Psi^*}{\partial\boldsymbol{\sigma}} = \Delta\lambda\mathbf{P}^*\boldsymbol{\sigma}^*. \tag{7}$$

For plane stress the in-plane relation between stresses and strains is derived from the 3D version using Hooke's law. By requiring that $\sigma_z, \sigma_{xz}, \sigma_{yz}$ vanish, and condensing the corresponding rows and columns from the elasticity matrix, the elastic in-plane stress-strain relation reduces to:

$$\boldsymbol{\sigma}^* = \mathbf{D}^e\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^e = (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{8}$$

within matrix notation $\boldsymbol{\sigma}^* = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, \boldsymbol{D}^e = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}, \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{bmatrix}, \boldsymbol{\varepsilon}^p = \begin{bmatrix} \epsilon_x^p \\ \epsilon_y^p \\ 2\epsilon_{xy}^p \end{bmatrix}$

and

$$\varepsilon_{xz} = \varepsilon_{yz} = 0, \varepsilon_z^e = -\frac{\nu}{1-\nu}\left(\varepsilon_x^e + \varepsilon_y^e\right). \tag{9}$$

Assuming that at step (n) all stresses and strains are known and satisfy equilibrium, and the next strain increment $\Delta\boldsymbol{\varepsilon}_{n+1}^*$ is known, then for the new stress in step (n+1) one can write

$$\boldsymbol{\sigma}_{n+1}^{*\mathrm{trial}} = \boldsymbol{\sigma}_n^* + \mathbf{D}^e\Delta\boldsymbol{\varepsilon}_{n+1}^*, \tag{10}$$

$$\boldsymbol{\sigma}_{n+1}^* = \boldsymbol{\sigma}_{n+1}^{*\mathrm{trial}} - \mathbf{D}^e\Delta\lambda\mathbf{P}^*\boldsymbol{\sigma}_{n+1}^*. \tag{11}$$

In Reference 1 this is rewritten to:

$$\boldsymbol{\sigma}_{n+1}^{*} = [\mathbf{C} + \Delta\lambda\mathbf{P}]^{-1}\mathbf{C}\boldsymbol{\sigma}_{n+1}^{*\text{trial}} \tag{12}$$

with

$$\mathbf{C} = (\mathbf{D}^{e})^{-1}. \tag{13}$$

In contrast to the scalar relation found for $\mathbf{s}_{n+1}$ in the 3D formulation, eq. (7.90) in Reference 1, now an inverse matrix equation for $\boldsymbol{\sigma}_{n+1}^{*}$ is obtained. When substituting (12) in the quadratic yield function (4), a single scalar nonlinear equation, having the incremental plastic multiplier $\Delta\lambda$ as the unknown, is obtained (eqs. 9.57–9.59 in Reference 1). The equation is reported to be nonlinear even in case of perfectly plastic material, opposite to its 3D counterpart. To solve this nonlinear equation for $\Delta\lambda$ a Newton Raphson procedure is used in Reference 1. Removing the out of plane stress components from the yield function apparently complicates the determination of the plastic strain increments and the return mapping. In the next sections is shown how the nonlinearity can be avoided.

2.3 | Consequences of plane stress projected equations

For plane stress a (in-plane) deviator stress is defined by De Souza Neto:[1]

$$\mathbf{s}^{*} = \mathbf{P}^{*}\boldsymbol{\sigma}^{*}, \tag{14}$$

$$\mathbf{s}^{*T} = \begin{bmatrix} s_x, & s_y, & s_{xy} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} (2\sigma_x - \sigma_y), & (2\sigma_y - \sigma_x), & 6\tau_{xy} \end{bmatrix}. \tag{15}$$

Note: The deviator stress (vector) defined by (14)[1] is confusing and different from the (tensor) definition (2) as here $s_{xy} = 2\tau_{xy}$.

For plane stress holds (with $\boldsymbol{\sigma}^{*T}$ according to (6)):

$$\boldsymbol{\sigma}^{*T}\mathbf{P}^{*}\boldsymbol{\sigma}^{*} = \boldsymbol{\sigma}^{*T}\mathbf{s}^{*} = \sigma_x s_x + \sigma_y s_y + \tau_{xy} s_{xy}, \tag{16}$$

or

$$\boldsymbol{\sigma}^{*T}\mathbf{P}^{*}\boldsymbol{\sigma}^{*} = s_x s_x + s_y s_y + \tau_{xy} s_{xy} + \frac{1}{3}(\sigma_x + \sigma_y)(s_x + s_y). \tag{17}$$

The Cauchy stress terms do not directly vanish. However by definition $s_x + s_y \equiv -s_z$,

Therefore in plane stress $s_x + s_y = -s_z = \frac{1}{3}(\sigma_x + \sigma_y)$ as $\sigma_z = 0$

Consequently

$$\frac{1}{3}(\sigma_x + \sigma_y)(s_x + s_y) = s_z s_z \tag{18}$$

and, as the out of plane shear stress components vanish in plane stress:

$$\boldsymbol{\sigma}^{*T}\mathbf{P}^{*}\boldsymbol{\sigma}^{*} = s_x s_x + s_y s_y + \tau_{xy} s_{xy} + s_z s_z = s_x s_x + s_y s_y + 2\tau_{xy}\tau_{xy} + s_z s_z = \mathbf{s} : \mathbf{s} = 2J_2. \tag{19}$$

Whereas the out-of-plane shear stress components vanish in plane stress, the deviator normal component s_z does not. Obviously in plane stress the values of the yield functions for plane stress and full 3D are identical:

$$\Psi^{*} = \frac{1}{2}\boldsymbol{\sigma}^{*T}\mathbf{P}^{*}\boldsymbol{\sigma}^{*} - \frac{1}{3}\sigma_{\text{yld}}^{2}\left(\epsilon^{p}\right) = \Psi = J_2 - \frac{1}{3}\sigma_{\text{yld}}^{2}\left(\epsilon^{p}\right). \tag{20}$$

The full 3D criterion in quadratic form can be expressed similar to the plane stress form (4)

$$\Psi = \frac{1}{2}\boldsymbol{\sigma}^{T}\mathbf{P}\boldsymbol{\sigma} - \frac{1}{3}\sigma_{\text{yld}}^{2}(\epsilon^{p}) = 0 \tag{21}$$

with

$$\mathbf{P}=\frac{1}{3}\begin{bmatrix}2 & -1 & -1 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ -1 & -1 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 6 & 0 & 0\\ 0 & 0 & 0 & 0 & 6 & 0\\ 0 & 0 & 0 & 0 & 0 & 6\end{bmatrix}. \tag{22}$$

In plane stress:

$$\boldsymbol{\sigma}^T=[\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{yz}]=[\sigma_x,\sigma_y,0,\tau_{xy},0,0] \tag{23}$$

and the deviatoric stress vector:

$$\tilde{\mathbf{s}}^T=(\mathbf{P}\boldsymbol{\sigma})^T=\frac{1}{3}\begin{bmatrix}(2\sigma_x-\sigma_y), & (2\sigma_y-\sigma_x), & (-\sigma_x-\sigma_y), & 6\tau_{xy}, & 0, & 0\end{bmatrix}=\begin{bmatrix}s_x, & s_y, & s_z, & s_{xy}, & 0, & 0\end{bmatrix}. \tag{24}$$

Observe that the in-plane deviator stress components in (15) are identical to the corresponding deviator stress component in (24), however, the nonvanishing deviator-component (s_z) is still present in (24) but is missing in (15) due to the plane stress projection. As a consequence of the plane stress projection, important information thus is lost, resulting in a more complicated nonlinear solution procedure.

For the Von Mises model replacing (1) by the plane stress projection (4) may be justified if it results in a more simple solution, but that is not the case as will be shown in the next section. Restricting to plane stress yield criteria may be required for more sophisticated sheet metal models that are expressed by in-plane stresses only, that is, Barlat et al.,[3] Yoshida et al.,[4] sheet metal models based on the Hill yield criterion Huetink et al.,[5] the Vegter yield criterion Vegter et al.[6] The latter model is based on biaxial sheet material tests.

2.4 | Closed form radial return algorithm for plane stress

There is no need to stick to Cauchy stresses in the yield function. Also for plane stress the yield function can be expressed in deviator stress components (19) and (20). Besides, as shown below, a linear equation for the plastic multiplier and an explicit constitutive function that satisfy the plane stress constraint can be obtained. Requiring that $\Psi^*=0$ at plastic deformation in plane stress may be replaced by requiring that $\Psi=0$, according to (20).

Note that the trial stress obtained by (10) satisfies the plane stress condition and is projected to the 3D stress vector (23):

$$\boldsymbol{\sigma}^{*T(\mathrm{trial})}=[\sigma_x,\sigma_y\tau_{xy}]^{\mathrm{trial}}\rightarrow\boldsymbol{\sigma}^{T(\mathrm{trial})}=[\sigma_x,\sigma_y,0,\tau_{xy},0,0]^{\mathrm{trial}} \tag{25}$$

and the deviator stress vector

$$\mathbf{s}^{T(\mathrm{trial})}=[s_x,s_y,s_z,\tau_{xy},0,0]^{\mathrm{trial}}=[s_x,s_y,-(s_x+s_y),\tau_{xy},0,0]^{\mathrm{trial}} \tag{26}$$

In order to be compatible with the full 3D formulation (as in Reference 1 (subsection 7.3.1)) a tensor representation is used in the following elaboration. Confusion about the definition of the deviator stress vector $\tilde{\mathbf{s}}$ (24) and the deviator stress tensor $\mathbf{s}$ (2) is then avoided. The components of a deviator stress tensor are:

$$[\mathbf{s}]=\begin{bmatrix}s_x & \tau_{xy} & 0\\ \tau_{xy} & s_y & 0\\ 0 & 0 & s_z\end{bmatrix}=\begin{bmatrix}s_x & \tau_{xy} & 0\\ \tau_{xy} & s_y & 0\\ 0 & 0 & -(s_x+s_y)\end{bmatrix}. \tag{27}$$

The plastic strain increment (now including the out-of-plane component) is:

$$\Delta \boldsymbol{\epsilon}^p = \Delta \lambda \frac{\partial \Psi}{\partial \boldsymbol{\sigma}} = \Delta \lambda \mathbf{s} \tag{28}$$

and

$$\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{\text{trial}} - 2G\Delta \boldsymbol{\epsilon}^p. \tag{29}$$

With substitution of (28):

$$\mathbf{s}_{n+1} + 2G\Delta\lambda \mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{\text{trial}} \tag{30}$$

or

$$\mathbf{s}_{n+1} = \frac{\mathbf{s}_{n+1}^{\text{trial}}}{(1+2G\Delta\lambda)}. \tag{31}$$

Now the plastic multiplier $\Delta\lambda$ is solved by requiring: $\Psi(s_{n+1}) = 0$

$$\rightarrow \Psi = \frac{1}{2}\left(\boldsymbol{\sigma}^T \mathbf{P} \boldsymbol{\sigma}\right)_{n+1} - \frac{1}{3}\sigma_{\text{yld}}^2\left(\epsilon_{n+1}^p\right) = \frac{1}{2}\mathbf{s}_{n+1} : \mathbf{s}_{n+1} - \frac{1}{3}\sigma_{\text{yld}}^2\left(\epsilon_{n+1}^p\right) = 0 \tag{32}$$

or, with (31),

$$\Psi = \frac{1}{2}\frac{\mathbf{s}_{n+1}^{\text{trial}} : \mathbf{s}_{n+1}^{\text{trial}}}{(1+2G\Delta\lambda)^2} - \frac{1}{3}\sigma_{\text{yld}}^2\left(\epsilon_{n+1}^p\right) = 0, \tag{33}$$

$$\frac{1}{2}\mathbf{s}_{n+1}^{\text{trial}} : \mathbf{s}_{n+1}^{\text{trial}} = \frac{1}{3}\sigma_{\text{yld}}^2\left(\epsilon_{n+1}^p\right)(1+2G\Delta\lambda)^2, \tag{34}$$

$$\sqrt{\frac{3}{2}\mathbf{s}_{n+1}^{\text{trial}} : \mathbf{s}_{n+1}^{\text{trial}}} = \sigma_{\text{yld}}\left(\epsilon_{n+1}^p\right)(1+2G\Delta\lambda), \tag{35}$$

and in case of ideal plasticity an explicit expression for the plastic multiplier is obtained:

$$\Delta\lambda = \frac{\sqrt{\frac{3}{2}\mathbf{s}_{n+1}^{\text{trial}} : \mathbf{s}_{n+1}^{\text{trial}}} - \sigma_{\text{yld}}}{2G\sigma_{\text{yld}}} \tag{36}$$

or

$$\Delta\lambda = \frac{\Phi^{\text{trial}}}{2G\sigma_{\text{yld}}}. \tag{37}$$

This can easily be extended to linearized hardening, see Appendix A.

Substitution of (37) into (31) gives the new deviator stress:

$$\mathbf{s}_{n+1} = \frac{\mathbf{s}_{n+1}^{\text{trial}}}{(1+2G\Delta\lambda)} = \frac{\mathbf{s}_{n+1}^{\text{trial}}}{\left(1+\frac{\Phi^{\text{trial}}}{\sigma_{\text{yld}}}\right)} \tag{38}$$

The in-plane Cauchy stress component can directly be calculated from the in-plane deviator components using the inverse of matrix $\mathbf{P}^*$:

$$\boldsymbol{\sigma}_{n+1}^* = \mathbf{P}^{*-1}\mathbf{s}_{n+1}^* \tag{39}$$

with

$$\mathbf{P}^{*-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}. \tag{40}$$

Note that matrix $\mathbf{P}^*$ is regular (contrary to its 3D counterpart $\mathbf{P}$) so the inverse exists. Be aware that here for $s_{xy(n+1)}$ should be substituted $s_{xy(n+1)} = 2\tau_{xy(n+1)}$ before applying (39), see also the note at (15).

Alternatively the new Cauchy stress can be calculated by substitution of (37) into (11)

$$\boldsymbol{\sigma}^*_{n+1} = \boldsymbol{\sigma}^{*\text{trial}}_{n+1} - \mathbf{D}^e \Delta\lambda \mathbf{s}^*_{n+1} = \boldsymbol{\sigma}^{*\text{trial}}_{n+1} - \mathbf{D}^e \frac{\Phi^{\text{trial}}}{2G\sigma_{\text{yld}}} \mathbf{s}^*_{n+1} \tag{41}$$

and

$$\Delta\boldsymbol{\varepsilon}^p = \Delta\lambda \mathbf{s}_{n+1} = \frac{\Phi^{\text{trial}}}{2G} \frac{\mathbf{s}^{\text{trial}}_{n+1}}{(\sigma_{\text{yld}} + \Phi^{\text{trial}})}. \tag{42}$$

3 | VERIFICATION

The closed form radial return algorithm for plane stress from Section 2.4 is implemented in Matlab. The response was compared to simple, square single-element ABAQUS simulations and a simulation with the open source Dieka package (http://www.dieka.org).

The results coincide very well, see Figure 1.

The the Abaqus mesh consist of one square plane stress element. The square Dieka mesh consist of four (triangular) plane stress elements, see Figure 2. All boundary nodal displacements are prescribed in both simulations in a way that $\varepsilon_{22} = -0.5\varepsilon_{11}$. The the mid-node displacements in the triangular mesh are the only unconstrained decrees of freedom.

The material properties are: Youngs Modulus: 210,000 MPa, Poisson's ratio: 0.3, Yield stress: 400 Mpa, (no hardening).

The simulation with 4 triangular elements is bases on a simplified plane stress version of the Hill yield criterion for plane stress sheet metal deformation and in-plane isotropy as described by Huetink et al.[5] The simplified Hill criterion

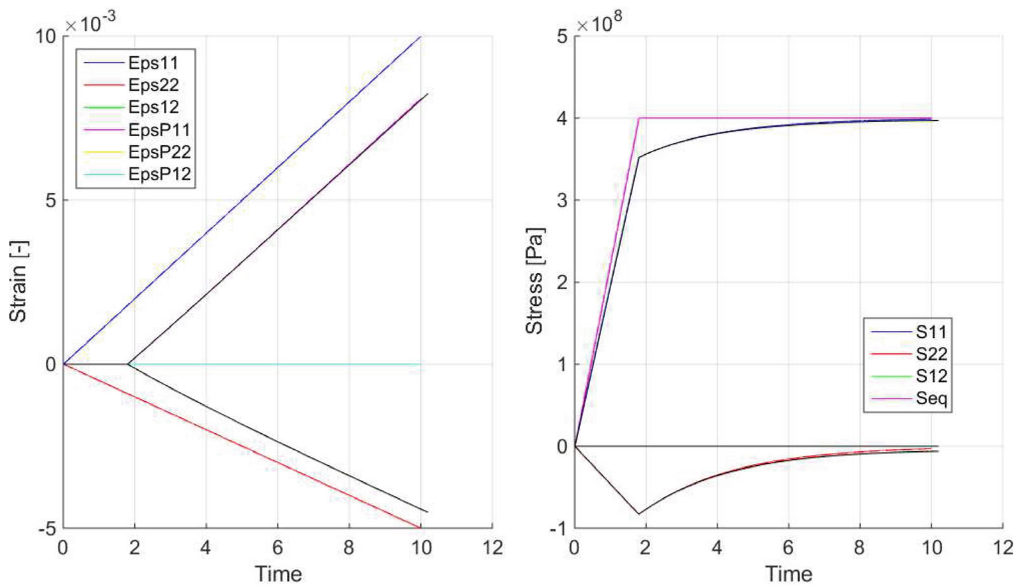

FIGURE 1 Predicted strains and stresses (solid lines), ABAQUS results (dashed lines) and Dieka results (black lines)

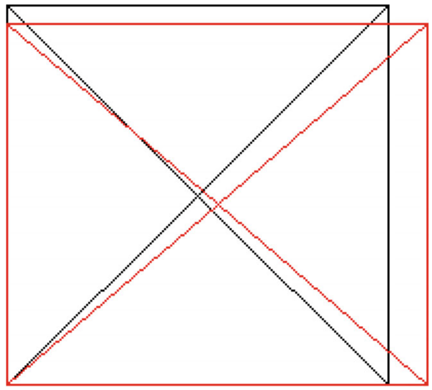

FIGURE 2 Triangular mesh, initial and deformed (exaggerated)

can then be written as:

$$\Phi^{\text{Hill}} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}^T \mathbf{P}^{\text{Hill}} \boldsymbol{\sigma}} - \sqrt{\frac{(R+1)}{2}} \left(\sigma_{\text{yld}}(\epsilon^p)\right) \tag{43}$$

with

$$\mathbf{P}^{\text{Hill}} = \frac{1}{3} \begin{bmatrix} (R+1) & -R & 0 \\ -R & (R+1) & 0 \\ 0 & 0 & 2(2R+1) \end{bmatrix}, \tag{44}$$

where R is the ratio of the transverse strain and the thickness strain in a tensile test in the in-plane x-direction: $R = \frac{\epsilon_y}{\epsilon_z}$. For verification $R = 1$, resulting in isotropy and coinciding with the method described by De Souza Neto et al.[1]

De Vries et al.[7] presented a Return Mapping algorithm for plane stress dedicated to thin walled pipes and vessels. Their approach is based on a 3D Von Mises yield criterion without a plane stress projection for the yield function. However, they introduced a constraint for the circumferential stress in the cylindrical pipe or vessel by the so called pressure vessel equation: $\sigma_\theta = \frac{(p^{\text{int}} - p^{\text{ext}})D}{2t}$ This constraint and the hardening model make their problem nonlinear.

4 | CONCLUSIONS

It is shown that in plane stress plasticity problems it is not required and may not be profitable to directly apply inclusion of the plane stress constraints into the three-dimensional elastic predictor and plastic corrector algorithm. It is sufficient to restrict the elastic predictor to plane stress and apply the unconstrained yield function.

Direct inclusion of the plane stress constraints into the yield function will lead to a nonlinear problem even in the case of an ideally plastic Von Mises criterion, whereas the unconstrained yield function result is a linear equation for the plastic multiplier. A Newton Raphson procedure can be avoided. The only real required constraint in plane stress is the calculation of the trial stress.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are openly available in huetink at https://research.utwente.nl/en/search/index/.

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APPENDIX . EXTENSION TO LINEARIZED HARDENING

$$\sigma_{\text{yld}}\left(\epsilon^p_{(n+1)}\right) = \sigma_{\text{yld}}\left(\epsilon^p_n\right) + \frac{d\sigma_{\text{yld}}}{d\epsilon_p}\Delta\epsilon^p = \sigma_{\text{yld}}\left(\epsilon^p_n\right) + h\Delta\epsilon^p \tag{A1}$$

with the equivalent plastic strain increment $\Delta\epsilon^p = \Delta\lambda\frac{2}{3}\sigma_{\text{yld}}$. The plastic multiplier (37) changes accordingly:

$$\Delta\lambda = \frac{3}{2}\frac{\Phi^{\text{trial}}}{(3G+h)\sigma_{\text{yld}}} \tag{A2}$$

The plastic multiplier obtained from the quadratic yield function differs from the square root formulation obtained in Reference 1(eq. (7.101)):

$$\Delta\gamma = \Delta\lambda\sigma_{\text{yld}}\frac{2}{3} = \frac{\Phi^{\text{trial}}}{3G+h}. \tag{A3}$$