Signal Processing for Phased Arrays

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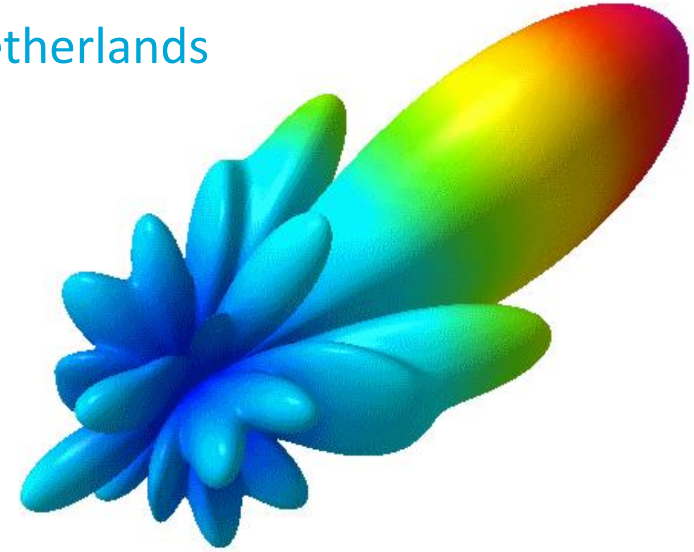

Outline

Part I – Introduction

- History and applications
- Advantages of array processing
- Phased array data model
  - Baseband signal
  - Narrowband signal
  - Single path and single user
  - Multipath and multiple users

Outline

Part II – Traditional Processing

- Beamforming
  - Matched filter
  - Zero forcing
  - Minimum mean square error
  - Minimum variance distortionless response
- Direction of arrival (DoA) estimation
  - Beamforming based methods
  - Multiple signal classification (MUSIC)
  - Estimation of signal parameters via rotational invariant techniques (ESPRIT)
  - Sparse reconstruction

Outline

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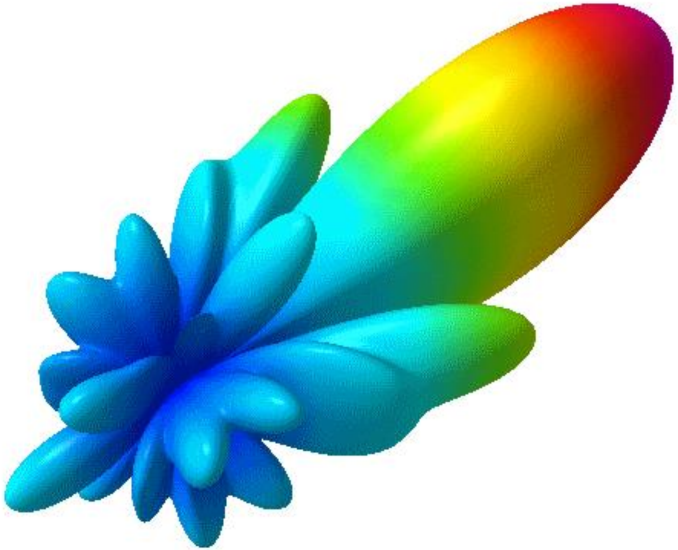
- Compressive array
  - Compressive sensing (CS)
  - CS in spatial domain
- Covariance based processing
  - Compressive covariance sensing (CCS)
  - CCS in spatial domain
  - CCS based array design
  - Virtual array principle
- Performance based array design

Part I:
Introduction

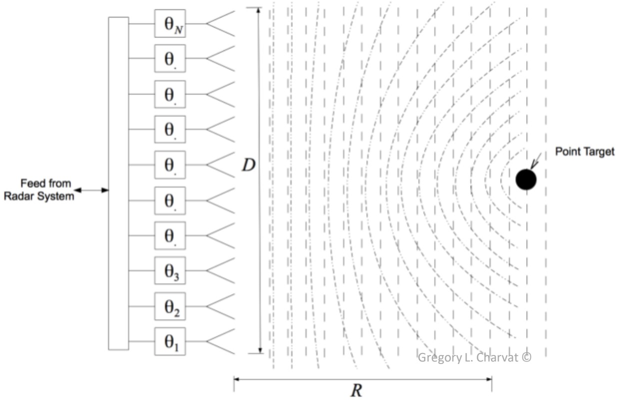


Introduction

- History and applications
- Advantages of array processing
- Phased array data model
  - Baseband signal
  - Narrowband signal
  - Single path and single user
  - Multipath and multiple users

History and Applications

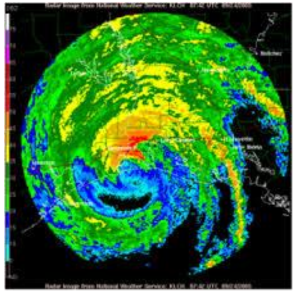
- "The quintessential goal of sensor array signal processing is the estimation of parameters by fusing temporal and spatial information, captured via sampling a wavefield with a set of judiciously placed antenna sensors."
  Two Decades of Array Signal Processing Research; Krim, Hamid & Viberg, Mats
- The sampling of wavefields arises in many applications



Radioastronomy

Ultrasound Imaging

MRI



Radar

Communications

History and Applications

Signal processing for phased arrays has long history – more than 50 years

- All started during the second world war with spatial filtering [Barlett, 1948]
- Classical time delay estimation methods enhanced spatial resolution
- Adaptivity was introduced [Capon, 1969] [Applebaum, 1976] [Buckley, 1986]
- Parametric estimation techniques extended the resolution limits of classical spatial filtering techniques [Burg, 1967] [MacDonald, 1969] [Ottersten, 1993]
- Subspace-based methods exploited the geometry of the data model [Roy, 1989] [Schmidt, 1981]
- Optimization-based methods allow for reduced measurements and resolution improvements [Malioutov et al, 2005]
- These methods are pervasive in many application domains

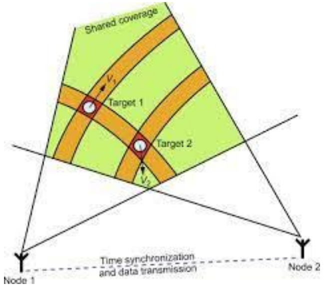
Advantages of Array Processing

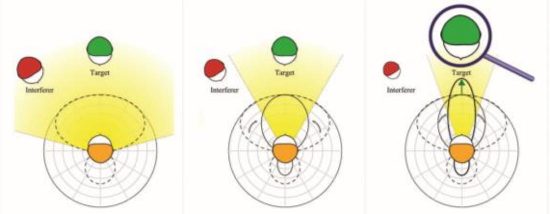
[Johnson-Dudgeon, 1993] [Krim-Viberg, 1996]

Improved performance

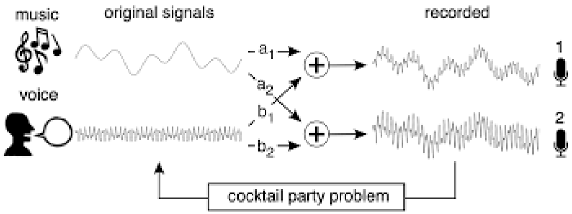
Diversity

Source localization





Interference reduction



Source separation

Phased Array Data Model

- We consider a passive radar / communication model
- Our data model will be based on
  - Baseband representation
  - Narrowband signals
- Extensions to wideband models are possible [van Veen-Buckley, 1988]
  - Space-time processing
  - Multiple frequencies using short-term Fourier transform

Baseband Representation

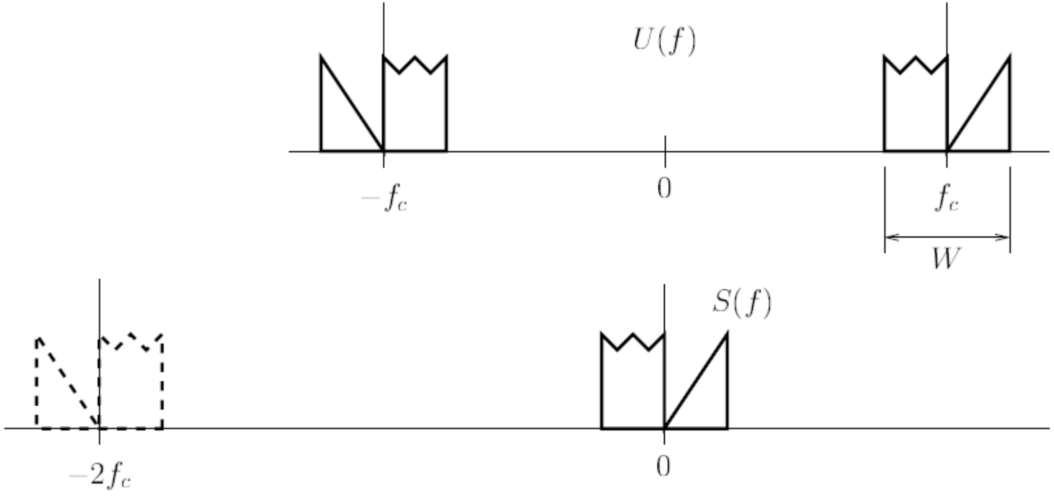
- An antenna receives a real-valued bandpass signal with frequency f_c

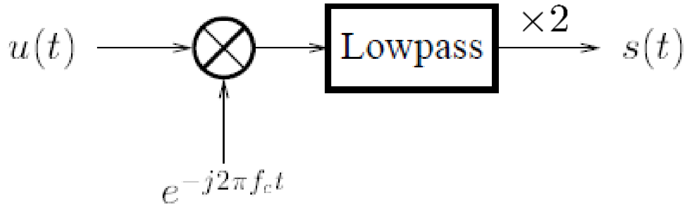
$$u(t) = \mathrm{real}\{s(t)e^{j2\pi f_c t}\} = x(t)\cos(2\pi f_c t) + y(t)\sin(2\pi f_c t)$$

- The baseband signal (or complex envelope) is

$$s(t) = x(t) + jy(t)$$

- The baseband signal $s(t)$ can be recovered from $u(t)$ by demodulation





Narrowband Signals

- Effect of small delays in $u(t)$ on the baseband signal $s(t)$

$$u_\tau(t) := u(t-\tau) = \mathrm{real}\{s(t-\tau)e^{-j2\pi f_c \tau}e^{j2\pi f_c t}\}$$

- The baseband version of the delayed signal is

$$s_\tau(t) = s(t-\tau)e^{-j2\pi f_c \tau}$$

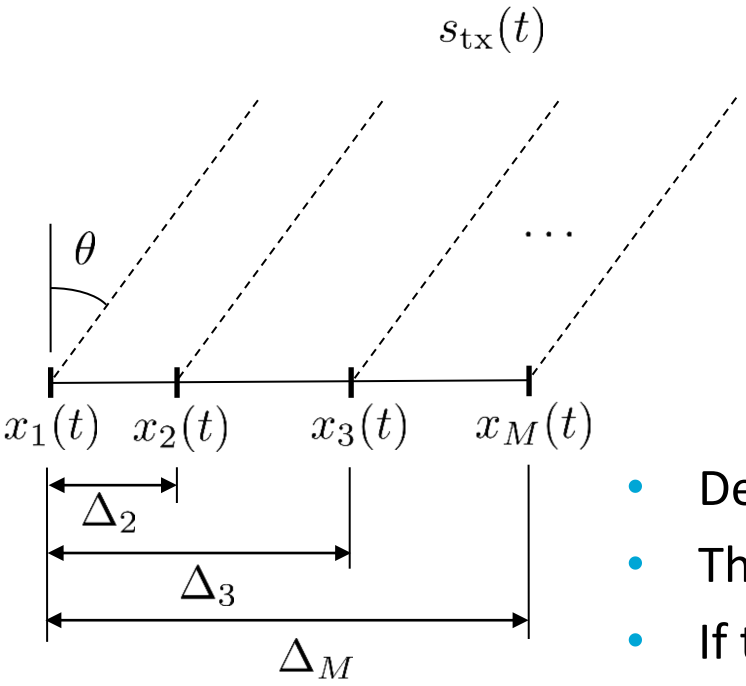
- Let W be the bandwidth of $s(t)$. If $e^{j2\pi f \tau} \approx 1$ for all $|f| \leq W/2$, then

$$s(t-\tau) = \int_{-W/2}^{W/2} S(f)e^{j2\pi f(t-\tau)}\mathrm{d}f \approx \int_{-W/2}^{W/2} S(f)e^{j2\pi f t}\mathrm{d}f = s(t)$$

$$\Rightarrow \quad s_\tau(t) \approx s(t)e^{-j2\pi f_c \tau}$$

For narrowband signals, short time delays amount to phase shifts in baseband

Data Model



- Far field assumption: planar waves
- $s_{\text{tx}}(t)$ is the transmitted signal
- θ is the direction of arrival
- A is the attenuation
- T_i is propagation time to i-th element

$$x_i(t) = As_{\text{tx}}(t - T_i)e^{-j2\pi f_c T_i}$$

- Defining $s(t) = s_{\text{tx}}(t - T_1),\ \tau_i = T_i - T_1,\ \beta = Ae^{-j2\pi f_c T_1}$
- Then we obtain $x_i(t) = \beta s(t - \tau_i)e^{-j2\pi f_c \tau_i}$
- If the delays over the array are small enough, then

$$\boxed{x_i(t) = \beta s(t)e^{-j2\pi f_c \tau_i}}$$

Data Model

- The delay τ_i can be expressed using θ and Δ_i (distance in wavelengths)

$$2\pi f_c \tau_i = -2\pi f_c \frac{\delta_i \sin\theta}{c} = -2\pi \frac{\delta_i}{\lambda} \sin\theta = -2\pi \Delta_i \sin\theta$$

- As a result we obtain

$$x_i(t) = \beta s(t) e^{j2\pi\Delta_i \sin\theta}$$

- Collecting the received signals into a vector leads to

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_M(t) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{j2\pi\Delta_2 \sin\theta} \\ \vdots \\ e^{j2\pi\Delta_M \sin\theta} \end{bmatrix} \beta s(t) = \boldsymbol{a}(\theta)\beta s(t)$$

array response vector

Data Model

- For a uniform linear array (ULA) where $\Delta_i = (i-1)\Delta$, we obtain

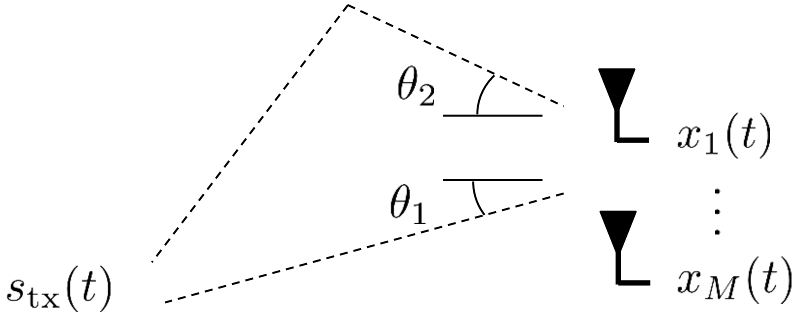
$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ e^{j2\pi\Delta\sin\theta} \\ \vdots \\ e^{j2\pi(M-1)\Delta\sin\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ \psi \\ \vdots \\ \psi^{M-1} \end{bmatrix} \qquad \psi = e^{j2\pi\Delta\sin\theta}$$

- This Vandermonde structure of the ULA can be exploited for DoA estimation
  - DoA estimation using ESPRIT (see later on)
- Non-uniform ULAs can be viewed as a compressed ULA
  - MUSIC and beamforming based DoA estimation possible (see later on)
  - Useful for covariance based array processing (see later on)

Data Model

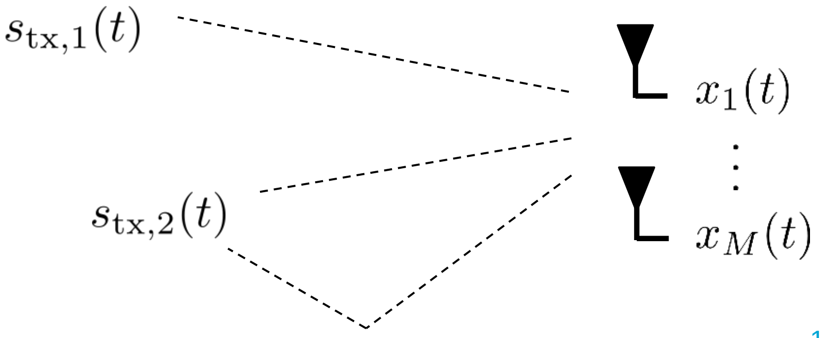
- Multipath

$$\boldsymbol{x}(t) = \boldsymbol{a}(\theta_1)\beta_1 s(t) + \boldsymbol{a}(\theta_2)\beta_2 s(t)$$
$$= [\boldsymbol{a}(\theta_1)\beta_1 + \boldsymbol{a}(\theta_2)\beta_2] s(t) = \boldsymbol{a}s(t)$$

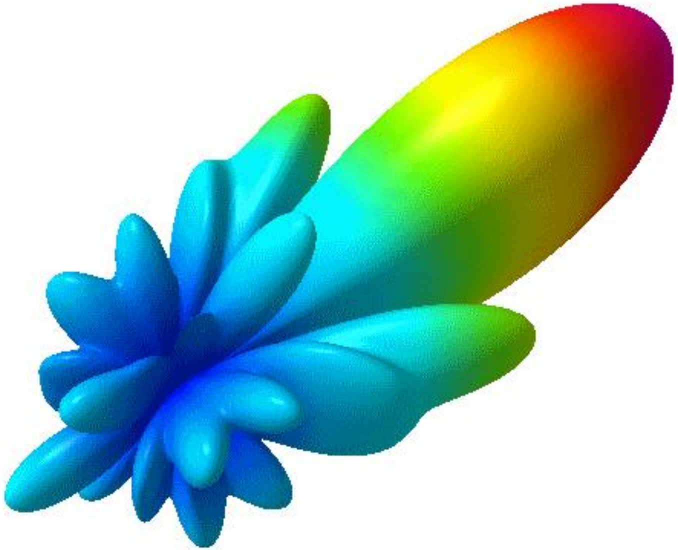


- Multiple users

$$\boldsymbol{x}(t) = \boldsymbol{a}_1 s_1(t) + \boldsymbol{a}_2 s_2(t)$$
$$= \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \boldsymbol{A}\boldsymbol{s}(t)$$



Part II:
Traditional Processing



Traditional Processing

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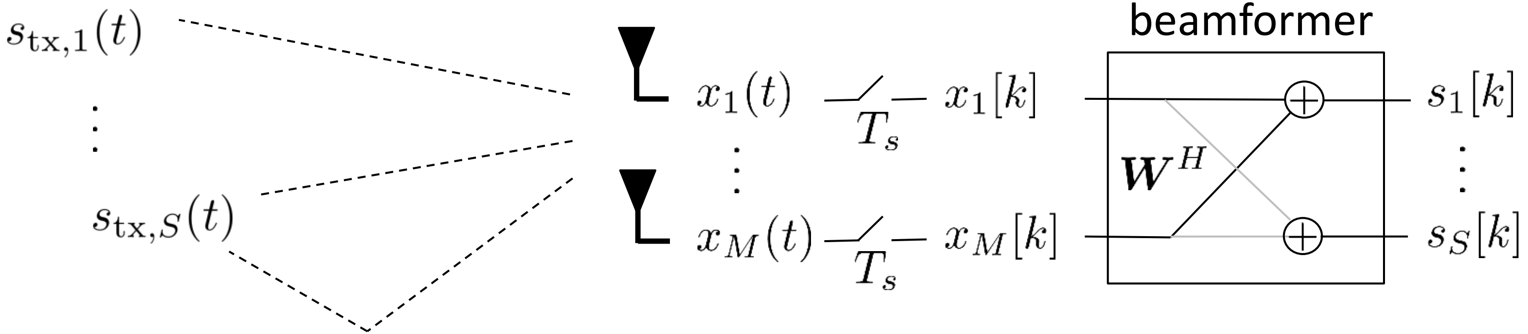
Beamforming [van Veen-Buckley, 1988]



- Assume S narrowband signals impinge on the array through multipath
- After sampling we obtain

$$\boldsymbol{x}[k] := \boldsymbol{x}(kT_s) = \sum_{i=1}^{S} \boldsymbol{a}_i s_i(kT_s) + \boldsymbol{n}(kT_s) := \sum_{i=1}^{S} \boldsymbol{a}_i s_i[k] + \boldsymbol{n}[k] = \boldsymbol{A}\boldsymbol{s}[k] + \boldsymbol{n}[k]$$

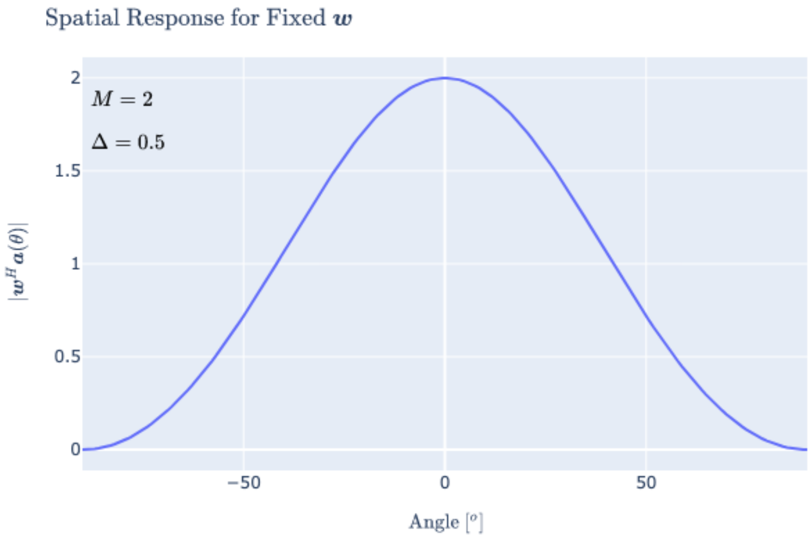
- The goal is to design a beamformer (BF) $\boldsymbol{W} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_S]$ such that

$$\boldsymbol{w}_i^H \boldsymbol{x}[k] = \hat{s}_i[k] \qquad \boldsymbol{W}^H \boldsymbol{x}[k] = \hat{\boldsymbol{s}}[k]$$

Array Response Graph

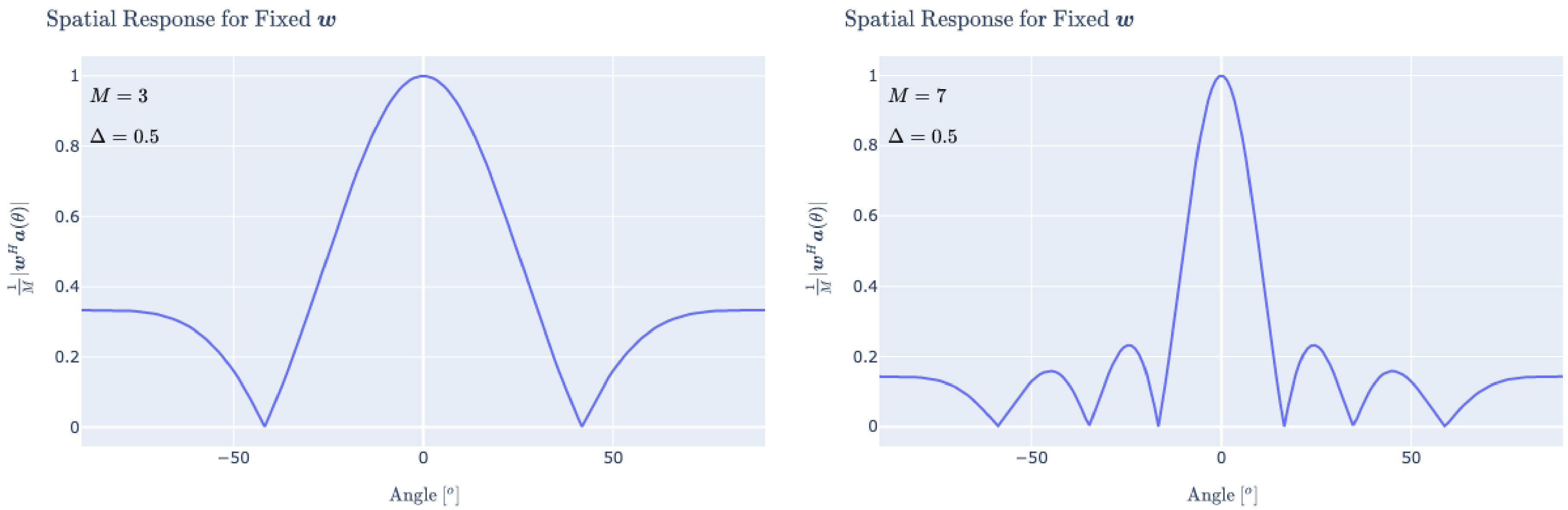
- Consider one user, one path and a fixed BF vector $\boldsymbol{w}$, e.g., $\boldsymbol{w} = \mathbf{1}$
- The output of the BF then is $y[k] = \boldsymbol{w}^H \boldsymbol{x}[k] = \boldsymbol{w}^H \boldsymbol{a}(\theta)\beta s[k]$
- The response to a unit-amplitude signal, i.e., $|\beta s[k]| = 1$, from direction θ is

$$|y[k]| = |\boldsymbol{w}^H \boldsymbol{a}(\theta)|$$



Array Response Graph

- With more antennas and same spacing, the resolution improves



Array Response Graph

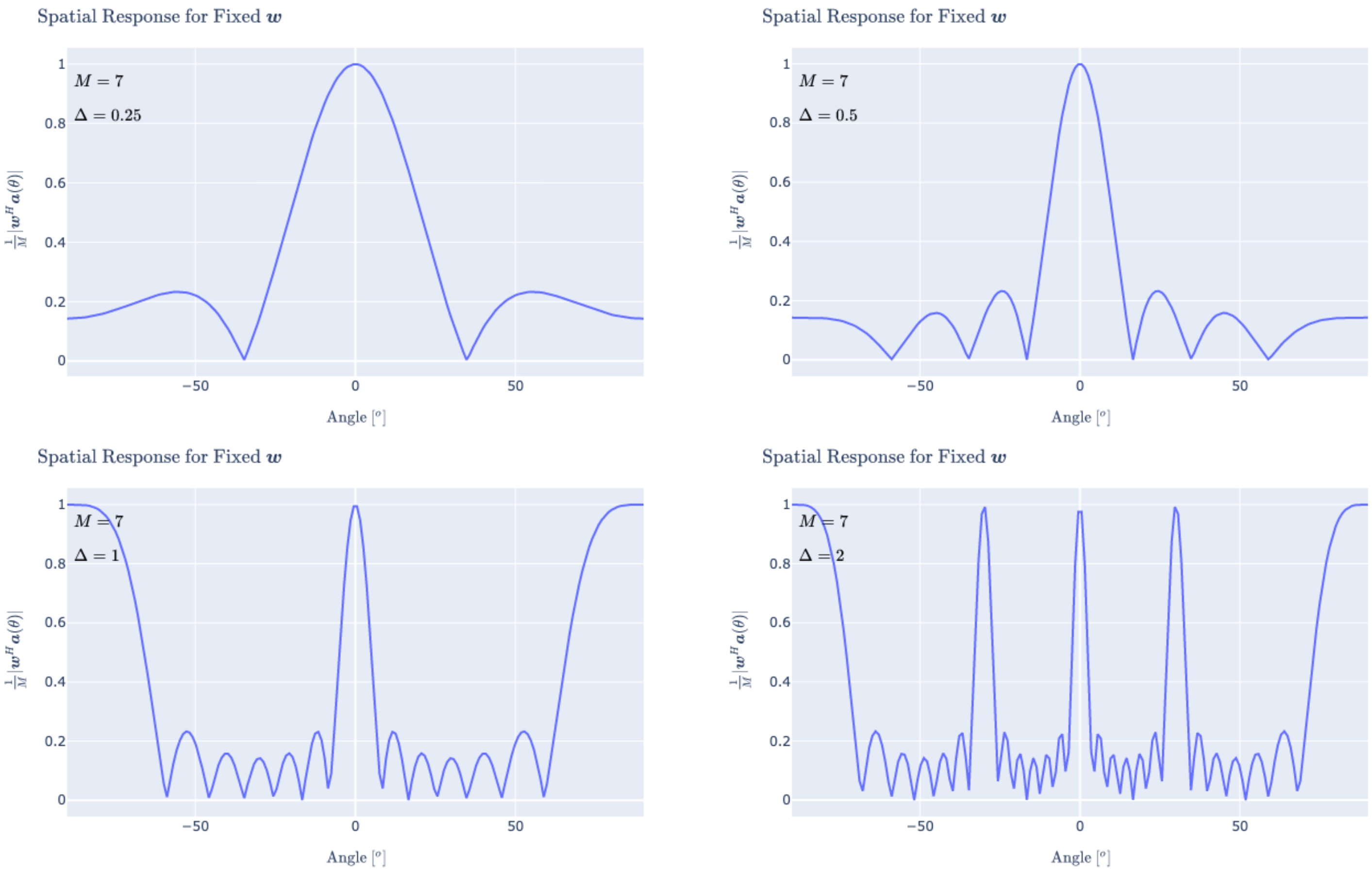
Ambiguity in the array response vector

- In a ULA we have

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ \psi \\ \vdots \\ \psi^{M-1} \end{bmatrix} \qquad \psi = e^{j2\pi\Delta\sin\theta}$$

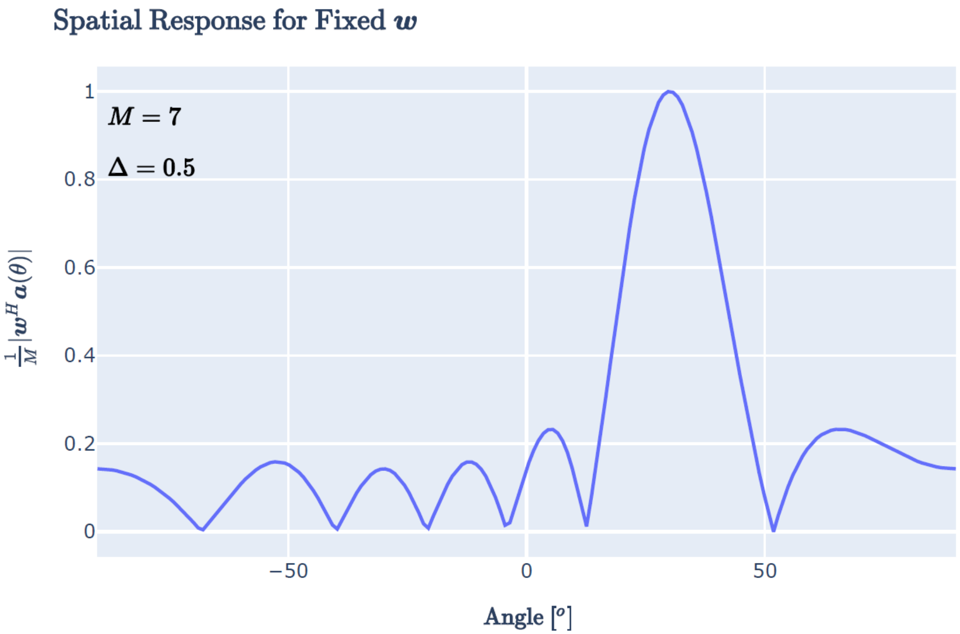
- Since $\sin\theta \in [-1, 1]$, we have $2\pi\Delta\sin\theta \in [-2\pi\Delta, 2\pi\Delta]$
- Hence, ψ determines θ uniquely iff $\Delta \leq 1/2$
- For $\Delta > 1/2$ there is spatial aliasing and grating lobes occur
- We can then still estimate $\boldsymbol{A}$ and do beamforming

Array Response Graph



Array Response Graph

- We can use other beamformers to steer the beam in different directions
- Consider for instance $\boldsymbol{w} = \boldsymbol{a}(30°)$ and look again at $|y[k]| = |\boldsymbol{w}^H \boldsymbol{a}(\theta)|$



Matched Filter Beamformer [Bartlett, 1948]

- Consider a single user in noise:

$$\boldsymbol{x}[k] = \boldsymbol{a}s[k] + \boldsymbol{n}[k], \quad \boldsymbol{R_n} = \mathrm{E}\{\boldsymbol{n}[k]\boldsymbol{n}[k]^H\} \quad \sigma_s^2 = \mathrm{E}\{|s[k]|^2\}$$

- The signal to noise ratio (SNR) after beamforming is given by

$$\mathrm{SNR} = \frac{|\boldsymbol{w}^H\boldsymbol{a}|^2\sigma_s^2}{\boldsymbol{w}^H\boldsymbol{R_n}\boldsymbol{w}} = \frac{\boldsymbol{w}^H(\sigma_s^2\boldsymbol{a}\boldsymbol{a}^H)\boldsymbol{w}}{\boldsymbol{w}^H\boldsymbol{R_n}\boldsymbol{w}}$$

- This is a generalized Rayleigh quotient which is maximized at

$$\boldsymbol{w} = \boldsymbol{R_n}^{-1}\boldsymbol{a}$$

- In case we have multiple users in noise, we then obtain

$$\boxed{\boldsymbol{W} = \boldsymbol{R_n}^{-1}\boldsymbol{A}}$$

- This is the matched filter BF, Bartlett BF, or maximum ratio combiner
- It is the beamformer that maximizes SNR

Zero Forcing Beamformer

- Consider multiple users in noise

$$\boldsymbol{x}[k] = \boldsymbol{A}\boldsymbol{s}[k] + \boldsymbol{n}[k], \quad \boldsymbol{R}_{\boldsymbol{n}} = \mathrm{E}\{\boldsymbol{n}[k]\boldsymbol{n}[k]^H\} \quad \boldsymbol{R}_{\boldsymbol{s}} = \mathrm{E}\{\boldsymbol{s}[k]\boldsymbol{s}[k]^H\}$$

- The BF for user i that forces interference to zero and mimizes noise energy is

$$\min_{\boldsymbol{w}_i} \boldsymbol{w}_i^H \boldsymbol{R}_{\boldsymbol{n}} \boldsymbol{w}_i \quad \text{s.t.} \quad \boldsymbol{w}_i^H \boldsymbol{A} = \boldsymbol{e}_i^H := [0, \ldots, 1, \ldots, 0]$$

- Using the technique of Lagrange multipliers, we obtain

$$\boldsymbol{w}_i = \boldsymbol{R}_{\boldsymbol{n}}^{-1} \boldsymbol{A} (\boldsymbol{A}^H \boldsymbol{R}_{\boldsymbol{n}}^{-1} \boldsymbol{A})^{-1} \boldsymbol{e}_i$$

- Stacking this for multiple users leads to

$$\boxed{\boldsymbol{W} = \boldsymbol{R}_{\boldsymbol{n}}^{-1} \boldsymbol{A} (\boldsymbol{A}^H \boldsymbol{R}_{\boldsymbol{n}}^{-1} \boldsymbol{A})^{-1}}$$

- This is the zero forcing (ZF) BF, or maximum likelihood BF
- It is the beamformer that maximizes SIR (and in that class maximizes SNR)

MMSE Beamformer

- Consider multiple users in noise

$$\boldsymbol{x}[k] = \boldsymbol{A}\boldsymbol{s}[k] + \boldsymbol{n}[k], \quad \boldsymbol{R}_{\boldsymbol{n}} = \mathrm{E}\{\boldsymbol{n}[k]\boldsymbol{n}[k]^H\} \quad \boldsymbol{R}_{\boldsymbol{s}} = \mathrm{E}\{\boldsymbol{s}[k]\boldsymbol{s}[k]^H\}$$

- The BF for user i that minimizes the output energy is given by

$$\min_{\boldsymbol{w}_i} \mathrm{E}\{|\boldsymbol{w}_i^H \boldsymbol{x}[k] - s_i[k]|^2\} = \min_{\boldsymbol{w}_i} \mathrm{E}\{|\boldsymbol{w}_i^H \boldsymbol{x}[k] - \boldsymbol{e}_i^H \boldsymbol{s}[k]|^2\}$$

- Setting the derivative towards $\boldsymbol{w}_i$ to zero, we obtain

$$\boldsymbol{w}_i = \boldsymbol{R}_{\boldsymbol{x}}^{-1} \boldsymbol{R}_{\boldsymbol{x}\boldsymbol{s}} \boldsymbol{e}_i \quad \boldsymbol{R}_{\boldsymbol{x}} = \mathrm{E}\{\boldsymbol{x}[k]\boldsymbol{x}[k]^H\} \quad \boldsymbol{R}_{\boldsymbol{x}\boldsymbol{s}} = \mathrm{E}\{\boldsymbol{x}[k]\boldsymbol{s}[k]^H\}$$

- Stacking this for multiple users leads to

$$\boxed{\boldsymbol{W} = \boldsymbol{R}_{\boldsymbol{x}}^{-1} \boldsymbol{R}_{\boldsymbol{x}\boldsymbol{s}} = (\boldsymbol{A}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{A}^H + \boldsymbol{R}_{\boldsymbol{n}})^{-1} \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{s}} \overset{\mathrm{MIL}}{=} \boldsymbol{R}_{\boldsymbol{n}}^{-1} \boldsymbol{A}(\boldsymbol{A}^H \boldsymbol{R}_{\boldsymbol{n}}^{-1} \boldsymbol{A} + \boldsymbol{R}_{\boldsymbol{s}}^{-1})^{-1}}$$

- This is the mimimum mean square error (MMSE) BF, or Wiener BF
- It is the beamformer that maximizes SINR

Discussion

- The ZF BF is equal to a MF BF followed by a decorrelator

$$\boldsymbol{W}_{\mathrm{ZF}} = \underbrace{\boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A}}_{\text{MF BF}} \; \underbrace{(\boldsymbol{A}^H\boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A})^{-1}}_{\text{decorrelator}}$$

- The MMSE BF is equal to a MF BF followed by a regularized decorrelator

$$\boldsymbol{W}_{\mathrm{MMSE}} = \underbrace{\boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A}}_{\text{MF BF}} \; \underbrace{(\boldsymbol{A}^H\boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A} + \boldsymbol{R}_{\boldsymbol{s}}^{-1})^{-1}}_{\text{regularized decorrelator}}$$

- If the noise approaches zero, the Wiener BF approaches the ZF BF

$$\boldsymbol{W}_{\mathrm{MMSE}} = \boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A} + \boldsymbol{R}_{\boldsymbol{s}}^{-1})^{-1} \quad \rightarrow \quad \boldsymbol{W}_{\mathrm{ZF}} = \boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{R}_{\boldsymbol{n}}^{-1}\boldsymbol{A})^{-1}$$

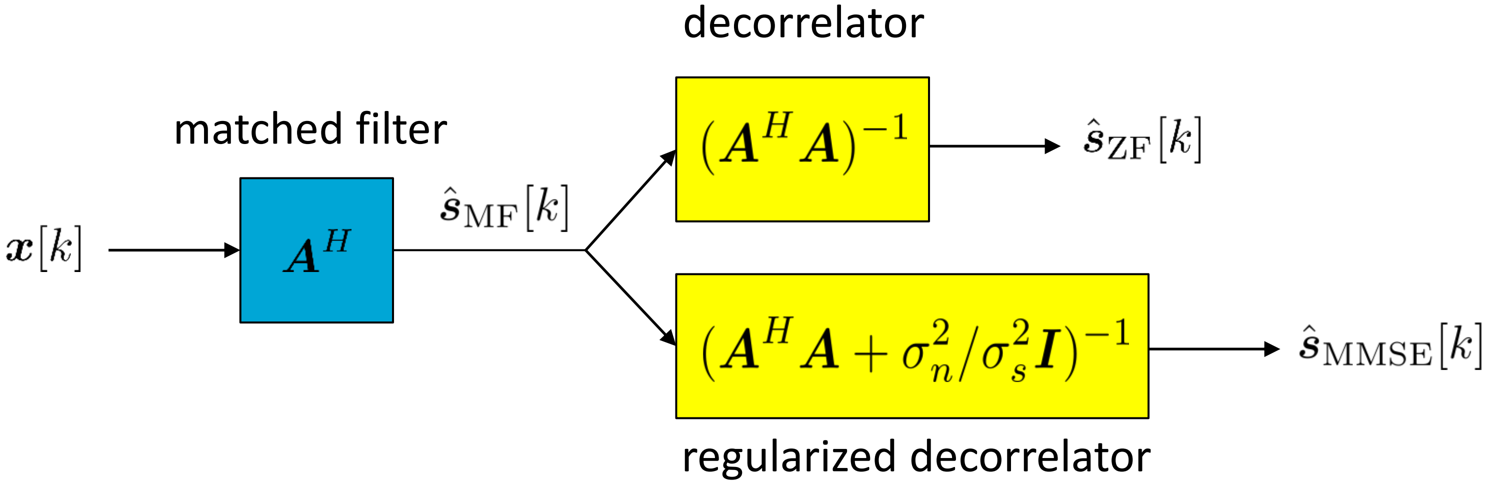
Discussion

- In case of uncorrelated users, $\boldsymbol{R_s} = \sigma_s^2 \boldsymbol{I}$, and white noise, $\boldsymbol{R_n} = \sigma_n^2 \boldsymbol{I}$:

$$\boldsymbol{W}_{\mathrm{MF}} = \boldsymbol{A}$$

$$\boldsymbol{W}_{\mathrm{ZF}} = \boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}$$

$$\boldsymbol{W}_{\mathrm{MMSE}} = (\boldsymbol{A}\boldsymbol{A}^H + \sigma_n^2/\sigma_s^2\boldsymbol{I})^{-1}\boldsymbol{A} \overset{\mathrm{MIL}}{=} \boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A} + \sigma_n^2/\sigma_s^2\boldsymbol{I})^{-1}$$



MVDR Beamformer [Capon, 1969]

- Consider again multiple users in noise
- Write the model as a single user system in noise plus interference

$$\boldsymbol{x}[k] = \boldsymbol{a}_i s_i[k] + \{\sum_{j=1,j\neq i}^{S} \boldsymbol{a}_j s_j[k] + \boldsymbol{n}[k]\}$$

- We can then force a fixed response towards user i by the constraint

$$\boldsymbol{w}_i^H \boldsymbol{a}_i = 1$$

- We can further minimize the output power, leading to

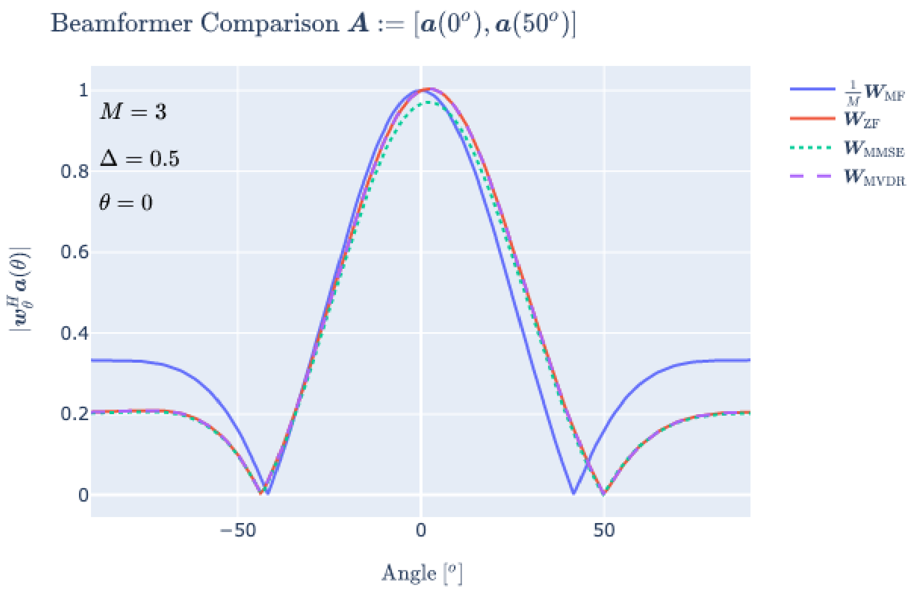
$$\min_{\boldsymbol{w}_i} \boldsymbol{w}_i^H \boldsymbol{R}_{\boldsymbol{x}} \boldsymbol{w}_i \quad \text{s.t.} \quad \boldsymbol{w}_i^H \boldsymbol{a}_i = 1$$

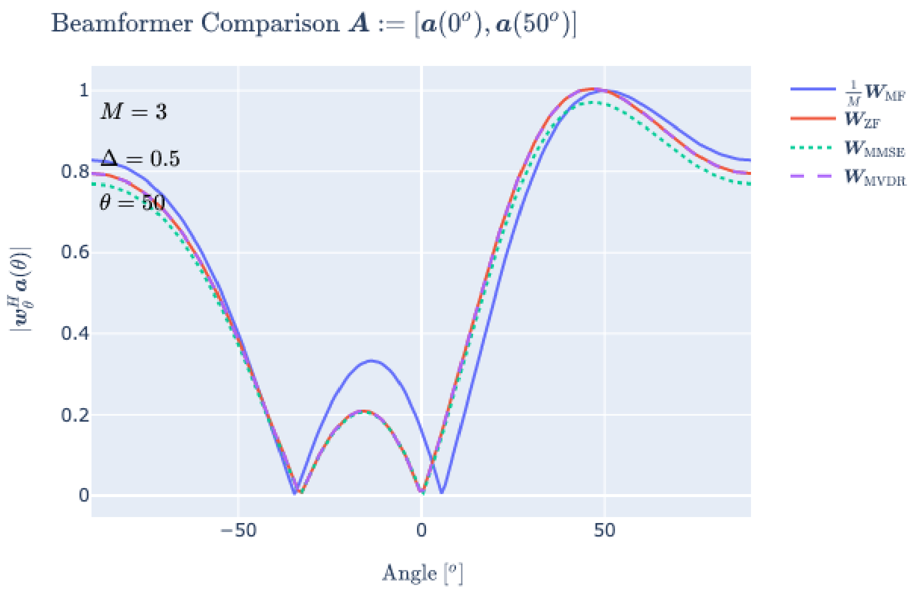
- Using the technique of Lagrange multipliers, we obtain

$$\boxed{\boldsymbol{w}_i = \boldsymbol{R}_{\boldsymbol{x}}^{-1} \boldsymbol{a}_i (\boldsymbol{a}_i^H \boldsymbol{R}_{\boldsymbol{x}}^{-1} \boldsymbol{a}_i)^{-1}}$$

- This is the mimimum variance distortionless response (MVDR) BF, or Capon BF
- It is a scaled (unbiased) version of the MMSE BF and maximizes the SINR

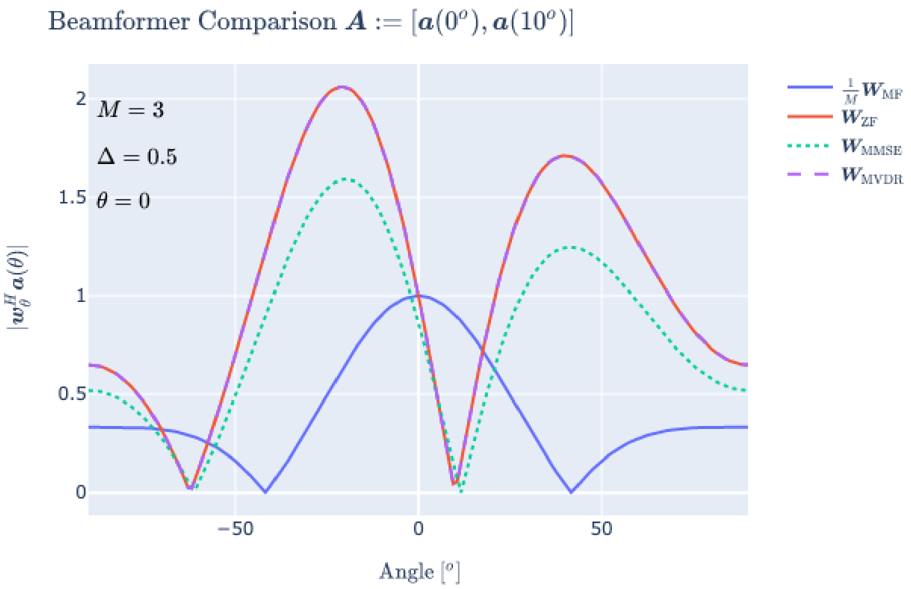
Beamformer Comparison

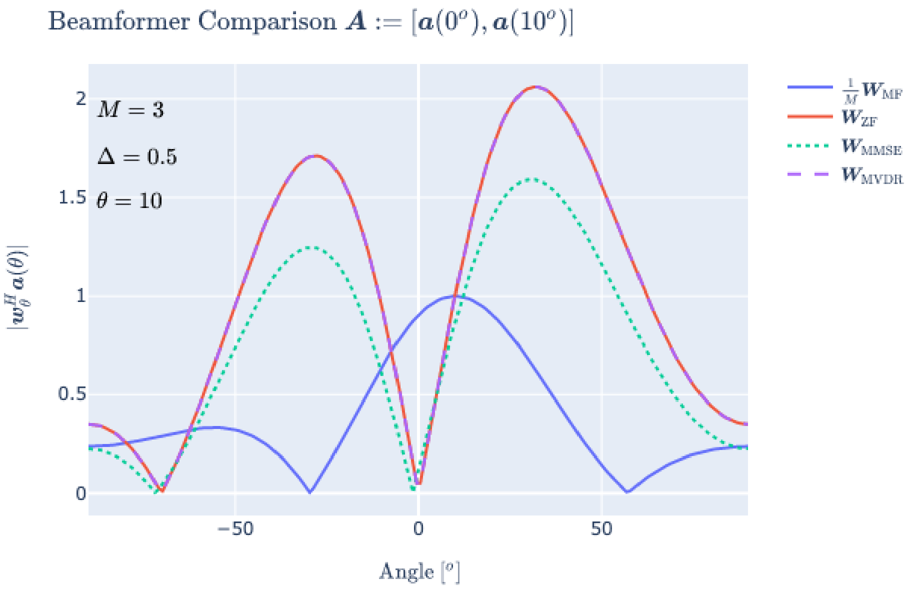




$$\boldsymbol{R}_s = \sigma_s^2 \boldsymbol{I} \qquad \frac{\sigma_s^2}{\sigma_n^2} = 10 \qquad \boldsymbol{R}_n = \sigma_n^2 \boldsymbol{I}$$

Beamformer Comparison





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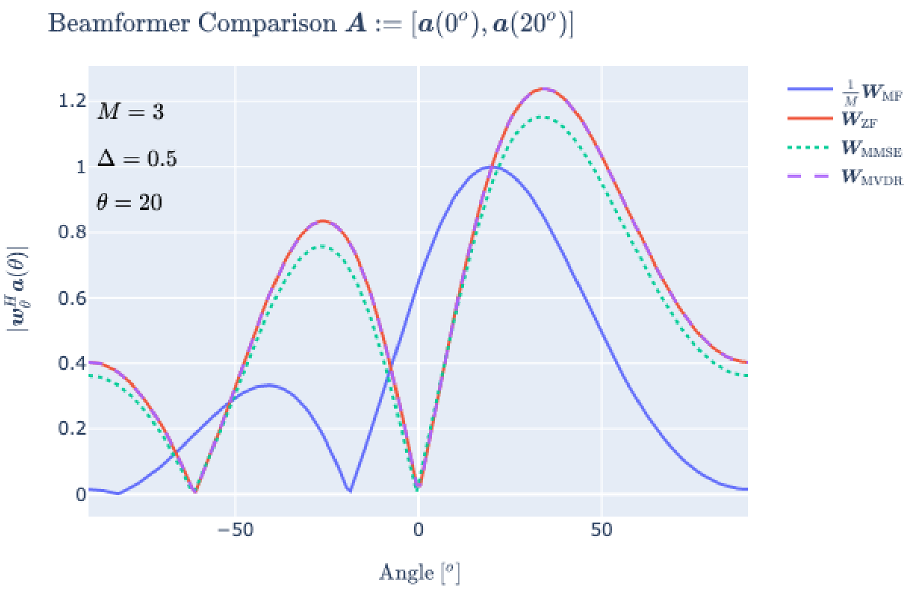

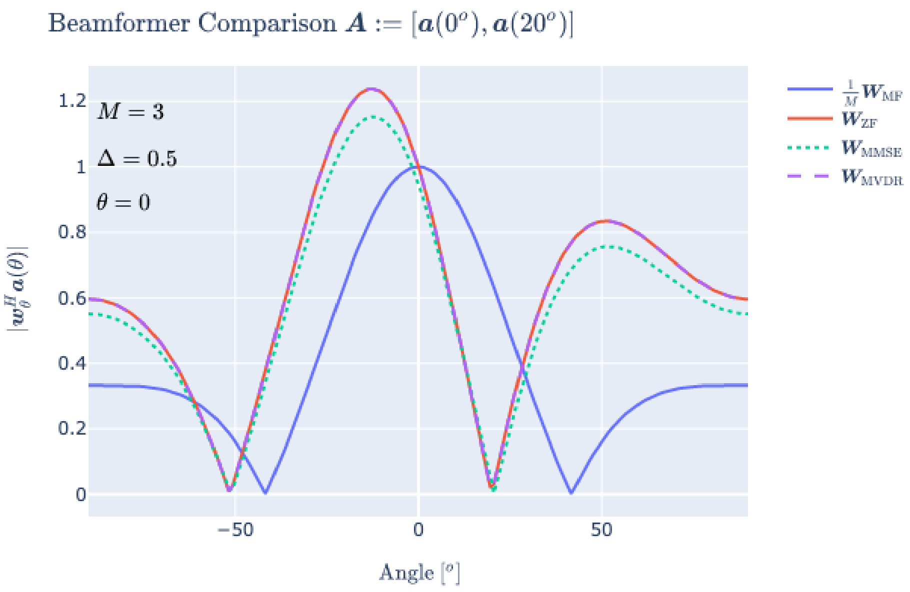

$$\boldsymbol{R}_s = \sigma_s^2 \boldsymbol{I} \qquad \frac{\sigma_s^2}{\sigma_n^2} = 10 \qquad \boldsymbol{R}_n = \sigma_n^2 \boldsymbol{I}$$

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Intuition Behind Direction Finding

- The array manifold is the curve that the vector $\boldsymbol{a}(\theta)$ describes when θ varies

$$\mathcal{A} = \{\boldsymbol{a}(\theta) : 0 \leq \theta \leq 2\pi\}$$

- The knowledge of $\mathcal{A}$ allows for direction finding
  - One source

$$\boldsymbol{x}(t) = \boldsymbol{a}(\theta)\beta s(t)$$

    For varying $s(t)$, the vector $\boldsymbol{x}(t)$ is confined to a line

  - Two sources

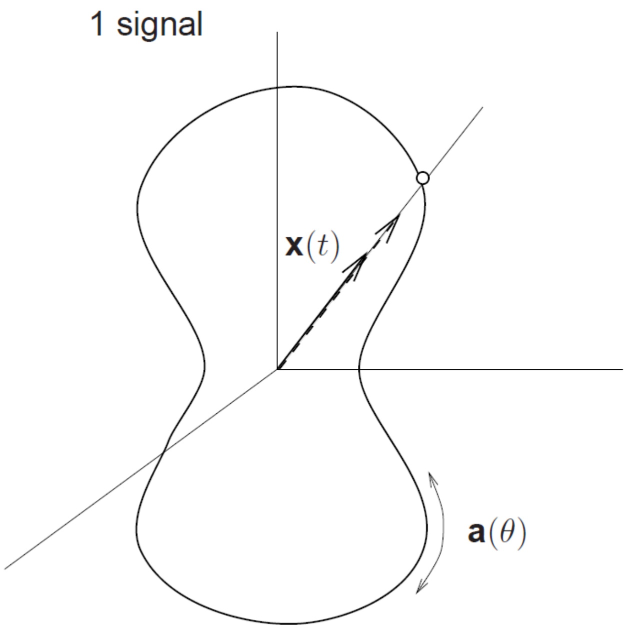
$$\boldsymbol{x}(t) = \boldsymbol{a}(\theta_1)\beta_1 s_1(t) + \boldsymbol{a}(\theta_2)\beta_2 s_2(t)$$

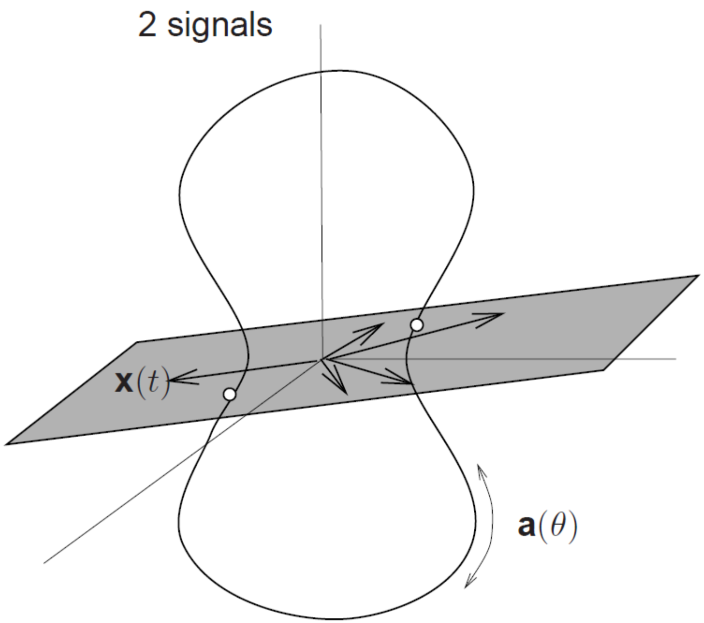
    For varying $s_1(t)$ and $s_2(t)$, the vector $\boldsymbol{x}(t)$ is now confined to a plane

  - The intersection of $\mathcal{A}$ with the line or plane results in the direction(s)

Intuition Behind Direction Finding

- Principle of direction finding





Intuition Behind Direction Finding

- This approach is not possible in case of multipath
- In that case, the array response vectors do not lie on the array manifold

$$\boldsymbol{a} = \sum_i \boldsymbol{a}(\theta_i)\beta_i$$

- Although it can be tackled, for now we consider a model without multipath

[Vanderveen et al, 1997]
[van der Veen et al, 1997]

$$\boldsymbol{x}[k] = \sum_{i=1}^{S} \boldsymbol{a}(\theta_i)\beta_i s_i[k] = \underbrace{[\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_S)]}_{\boldsymbol{A}(\boldsymbol{\theta})} \underbrace{\begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_S \end{bmatrix}}_{\boldsymbol{B}(\boldsymbol{\beta})} \underbrace{\begin{bmatrix} s_1[k] \\ \vdots \\ s_S[k] \end{bmatrix}}_{\boldsymbol{s}[k]}$$

$$\boxed{\boldsymbol{x}[k] = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{B}(\boldsymbol{\beta})\boldsymbol{s}[k]}$$

Maximum likelihood DoA Estimation

- Combining multiple snapshots we obtain

$$\boldsymbol{X} = [\boldsymbol{x}[1], \ldots, \boldsymbol{x}[K]] = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{B}(\boldsymbol{\beta})[\boldsymbol{s}[1], \ldots, \boldsymbol{s}[K]] = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{B}(\boldsymbol{\beta})\boldsymbol{S}$$

- The simplest way to formulate the DoA estimation problem is as

$$\boxed{\min_{\boldsymbol{\theta},\boldsymbol{\beta},\boldsymbol{S}} \|\boldsymbol{X} - \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{B}(\boldsymbol{\beta})\boldsymbol{S}\|_F^2}$$

- This is a maximum likelihood formulation in case of Gaussian noise
- The problem is very hard to tackle
  - Alternating minimization that might get stuck in a local minimum
    - $\Rightarrow$ Can be solved by assuming some training symbols
  - Complicated multi-dimensional search for $\boldsymbol{\theta}$

Beamforming Based Methods

- Design a beamformer $\boldsymbol{w}$ for a specific direction θ, i.e., $\boldsymbol{w}(\theta)$
- Scan all directions and maximize the output energy of the beamformer

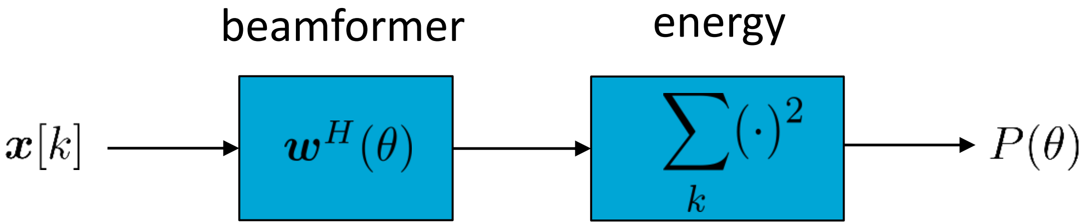
deterministic

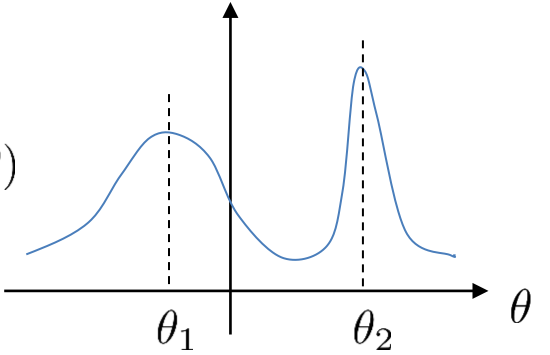
$$\max_{\theta} \|\boldsymbol{w}^H(\theta)\boldsymbol{X}\|_2^2$$

stochastic

$$\max_{\theta} \boldsymbol{w}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}\boldsymbol{w}(\theta)$$

  - For a single user, this resembles the array response graph of the BF
  - For multiple users, choose the S largest maxima





Matched Filter Based DoA Estimation

- Here it is assumed that the noise color is not known and hence

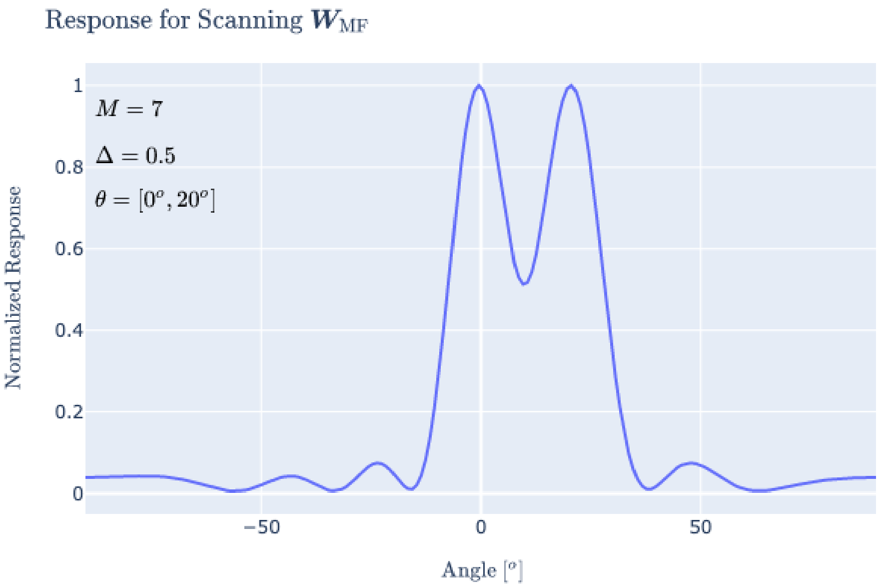
$$\boldsymbol{w}(\theta) = \boldsymbol{a}(\theta)$$

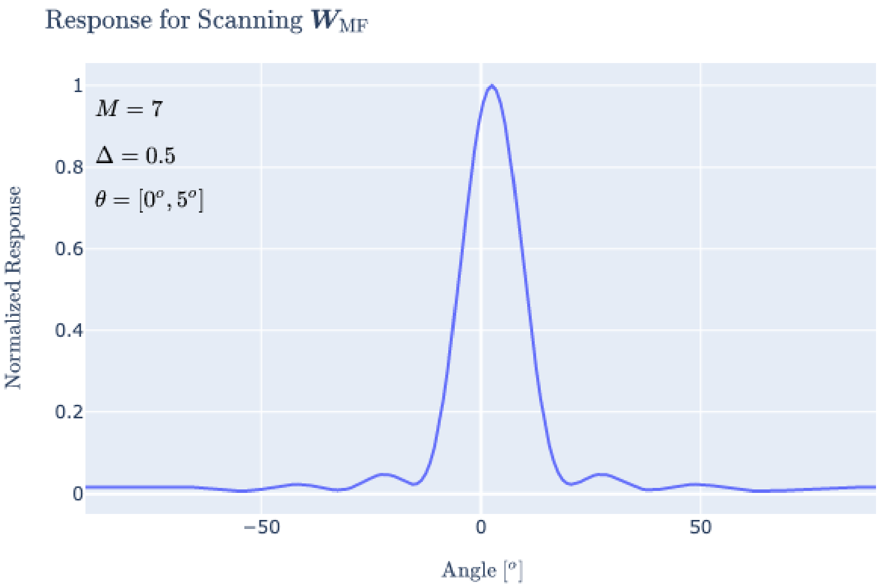
- The DoA estimation method then becomes

$$\boxed{\max_{\theta} \boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}\boldsymbol{a}(\theta)}$$

- If needed a normalization can be done with $||\boldsymbol{a}(\theta)||^2$
- For a single user in white noise, the considered matched filter is optimal
- For multiple users, the matched filter is not optimal
  - Users need to be well-separated
  - Biased estimates are generally obtained

Matched Filter Based DoA Estimation





MVDR Based DoA Estimation

- A more accurate beamformer is given by the MVDR BF

$$\boldsymbol{w}(\theta) = \boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta)(\boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta))^{-1}$$

- Computing the output power of this BF leads to

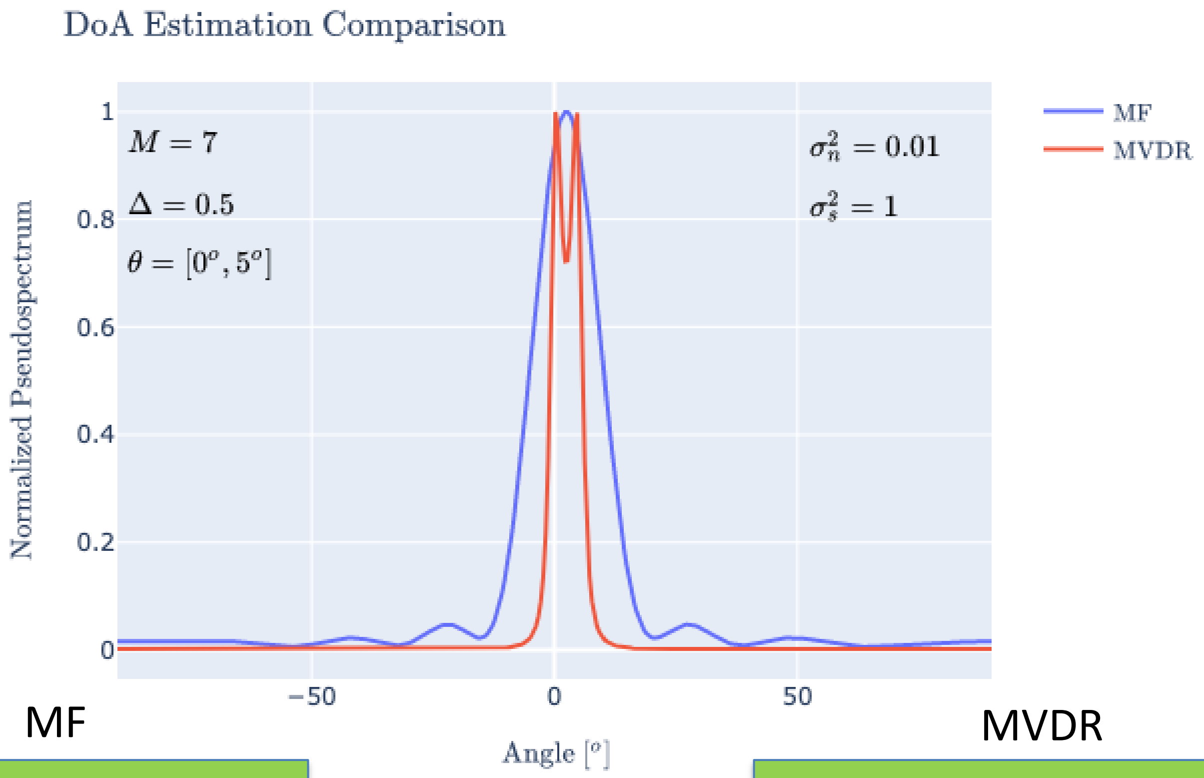
$$\begin{aligned}\boldsymbol{w}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}\boldsymbol{w}(\theta) &= \{(\boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta))^{-1}\boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}^{-1}\}\boldsymbol{R}_{\boldsymbol{x}}\{\boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta)(\boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta))^{-1}\} \\ &= (\boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta))^{-1}\end{aligned}$$

- The DoA estimation method then becomes

$$\boxed{\max_{\theta}(\boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta))^{-1}}$$

- Leads to a much better separation of the users

MVDR Based DoA Estimation



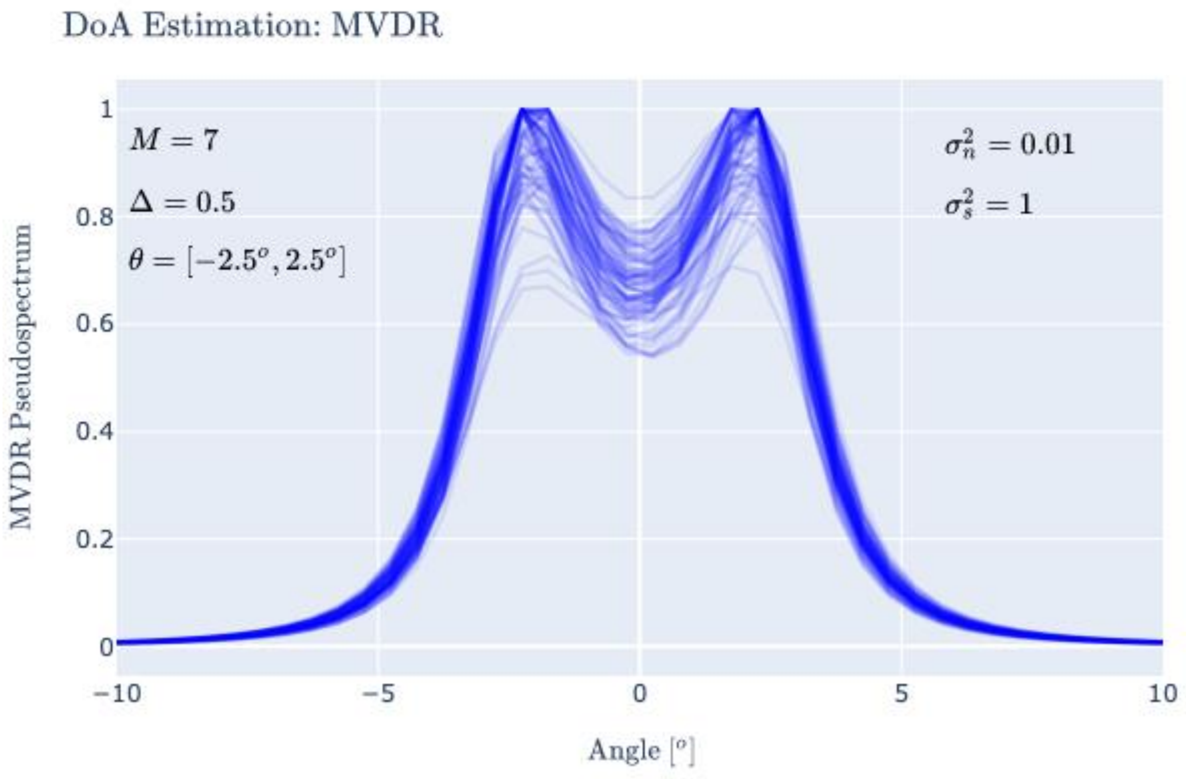
MF

$$\max_{\theta} \boldsymbol{a}^H(\theta)\boldsymbol{R_x}\boldsymbol{a}(\theta)$$

MVDR

$$\max_{\theta} (\boldsymbol{a}^H(\theta)\boldsymbol{R_x}^{-1}\boldsymbol{a}(\theta))^{-1}$$

MVDR Based DoA Estimation



#Snapshots: 100

Result of multiple realizations

MUSIC [Schmidt, 1981]

- This is a singular value (eigenvalue) decomposition-based technique
  - Deterministic (noiseless): $\boldsymbol{X} = \boldsymbol{A}\boldsymbol{B}\boldsymbol{S} = \boldsymbol{U}_s\boldsymbol{\Sigma}_s\boldsymbol{V}_s^H + \boldsymbol{U}_n\boldsymbol{0}\boldsymbol{V}_n^H$
  - Stochastic (white noise): $$\begin{aligned}\boldsymbol{R}_x &= \boldsymbol{A}\boldsymbol{B}\boldsymbol{R}_s\boldsymbol{B}^H\boldsymbol{A}^H + \sigma_n^2\boldsymbol{I} \\ &= \boldsymbol{U}_s(\boldsymbol{\Lambda}_s + \sigma_n^2\boldsymbol{I})\boldsymbol{U}_s^H + \boldsymbol{U}_n(\sigma_n^2\boldsymbol{I})\boldsymbol{U}_n^H\end{aligned}$$
- The SVD (EVD) reveals a relation between $\boldsymbol{U}_s$ or $\boldsymbol{U}_n$ and $\boldsymbol{A}(\boldsymbol{\theta})$

$$\mathrm{span}\{\boldsymbol{U}_s\} = \mathrm{span}\{\boldsymbol{A}(\boldsymbol{\theta})\}, \quad \mathrm{span}\{\boldsymbol{U}_n\} \perp \mathrm{span}\{\boldsymbol{A}(\boldsymbol{\theta})\}, \quad \boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_S)]$$

- We select $\{\theta_i\}_{i=1}^S$ to make $\boldsymbol{A}(\boldsymbol{\theta})$ *fit* $\mathrm{span}\{\boldsymbol{U}_s\}$ or *misfit* $\mathrm{span}\{\boldsymbol{U}_n\}$
- This can be done per angle and does not lead to a multi-dimensional search

$$\boxed{\boldsymbol{U}_n^H\boldsymbol{a}(\theta_i) = \boldsymbol{0}, \quad i = 1, 2, \ldots, S}$$

MUSIC

- The DoA estimation method then becomes

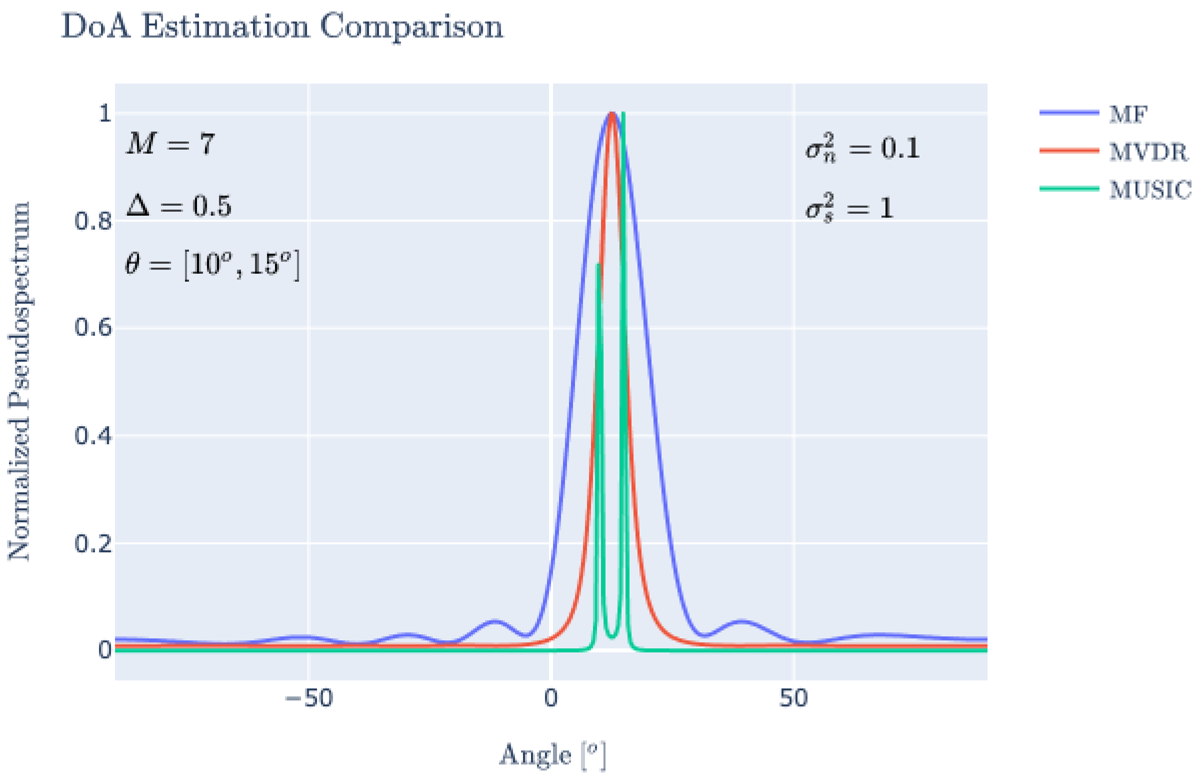
$$\min_{\theta} \|\boldsymbol{a}^H(\theta)\boldsymbol{U}_n\|^2 = \min_{\theta} \boldsymbol{a}^H(\theta)\boldsymbol{U}_n\boldsymbol{U}_n^H\boldsymbol{a}(\theta)$$

- Similar to BF based methods, we can also maximize the (*pseudo)* spectrum

$$\max_{\theta} (\boldsymbol{a}^H(\theta)\boldsymbol{U}_n\boldsymbol{U}_n^H\boldsymbol{a}(\theta))^{-1}$$

- If number of sources is smaller than number of sensors, $S < M$, then
  - Exact DoAs in noiseless deterministic case: $SNR \rightarrow \infty$
  - Exact DoAs in stochastic case with white noise: $K \rightarrow \infty$
- Estimates are *statistically consistent*

MUSIC



MF

$$\max_{\theta} \boldsymbol{a}^H(\theta)\boldsymbol{R_x}\boldsymbol{a}(\theta)$$

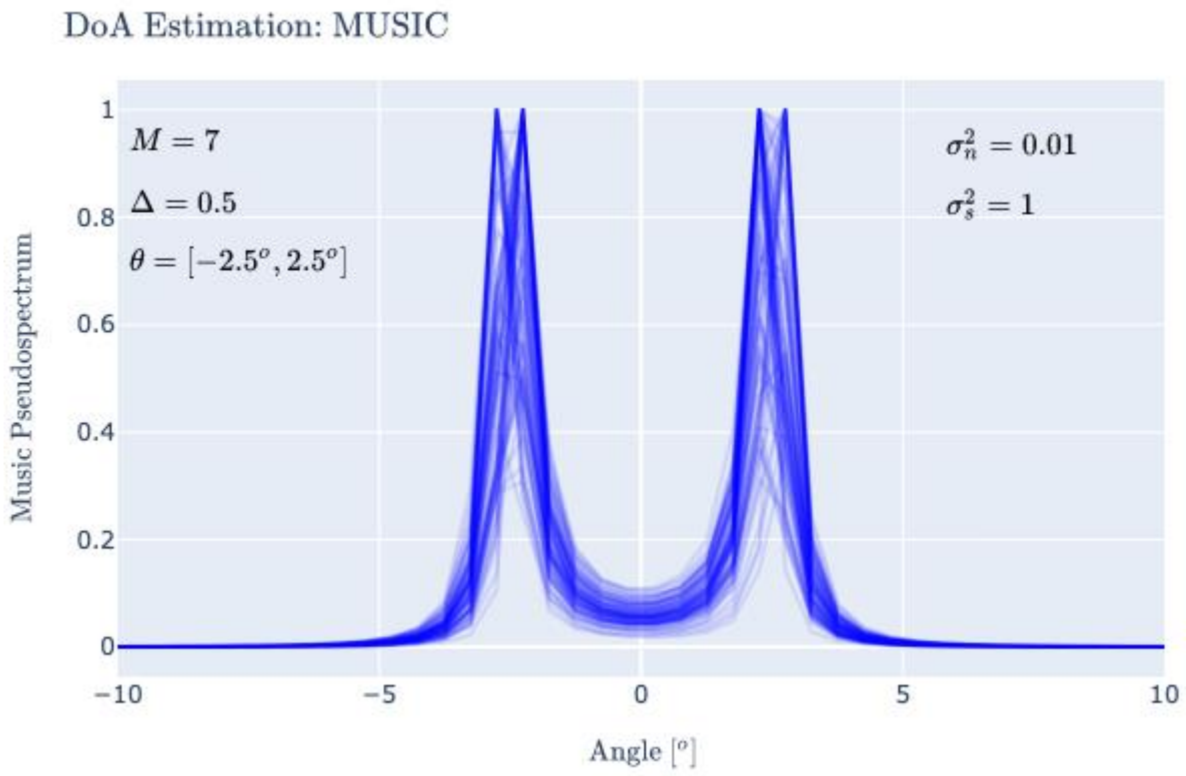
MVDR

$$\max_{\theta}(\boldsymbol{a}^H(\theta)\boldsymbol{R}_{\boldsymbol{x}}^{-1}\boldsymbol{a}(\theta))^{-1}$$

MUSIC

$$\max_{\theta}(\boldsymbol{a}^H(\theta)\boldsymbol{U}_n\boldsymbol{U}_n^H\boldsymbol{a}(\theta))^{-1}$$

MUSIC



#Snapshots: 100

Result of multiple realizations

ESPRIT [Roy-Kailath, 1989]

- Also an SVD (EVD)-based technique yet relying on a ULA
- For a ULA we obtain the shift-invariance property

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ e^{j2\pi\Delta\sin\theta} \\ \vdots \\ e^{j2\pi(M-1)\Delta\sin\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ \psi \\ \vdots \\ \psi^{M-1} \end{bmatrix} \quad \boldsymbol{a}_{\mathrm{t}}(\theta) \quad \boldsymbol{a}_{\mathrm{b}}(\theta)$$

$$\psi = e^{j2\pi\Delta\sin\theta}$$

$$\boldsymbol{a}_{\mathrm{t}}(\theta) = \begin{bmatrix} 1 \\ \psi \\ \vdots \\ \psi^{M-2} \end{bmatrix}, \quad \boldsymbol{a}_{\mathrm{b}}(\theta) = \begin{bmatrix} \psi \\ \psi^2 \\ \vdots \\ \psi^{M-1} \end{bmatrix}$$

$$\boxed{\boldsymbol{a}_{\mathrm{b}}(\theta) = \boldsymbol{a}_{\mathrm{t}}(\theta)\psi}$$

ESPRIT

- Using this property, let us combine the first and last $M-1$ antennas

$$\boldsymbol{x}_{\mathrm{t}}[k] = \begin{bmatrix} x_1[k] \\ \vdots \\ x_{M-2}[k] \end{bmatrix}, \quad \boldsymbol{x}_{\mathrm{b}}[k] = \begin{bmatrix} x_2[k] \\ \vdots \\ x_{M-1}[k] \end{bmatrix}, \qquad \begin{aligned} \boldsymbol{X}_{\mathrm{t}} &= [\boldsymbol{x}_{\mathrm{t}}[1], \ldots, \boldsymbol{x}_{\mathrm{t}}[K]] \\ \boldsymbol{X}_{\mathrm{b}} &= [\boldsymbol{x}_{\mathrm{b}}[1], \ldots, \boldsymbol{x}_{\mathrm{b}}[K]] \end{aligned}$$

- We then obtain

$$\boldsymbol{x}_{\mathrm{t}}[k] = \textstyle\sum_{i=1}^{S} \boldsymbol{a}_{\mathrm{t}}(\theta_i)\beta_i s_i[k] \qquad \Rightarrow \quad \boxed{\boldsymbol{X}_{\mathrm{t}} = \boldsymbol{A}_{\mathrm{t}}\boldsymbol{B}\boldsymbol{S}}$$

$$\boldsymbol{x}_{\mathrm{b}}[k] = \textstyle\sum_{i=1}^{S} \boldsymbol{a}_{\mathrm{b}}(\theta_i)\beta_i s_i[k] = \sum_{i=1}^{S} \boldsymbol{a}_{\mathrm{t}}(\theta_i)\psi_i\beta_i s_i[k] \qquad \Rightarrow \quad \boxed{\boldsymbol{X}_{\mathrm{b}} = \boldsymbol{A}_{\mathrm{t}}\boldsymbol{\Psi}\boldsymbol{B}\boldsymbol{S}}$$

$$\boldsymbol{A}_{\mathrm{t}} = [\boldsymbol{a}_{\mathrm{t}}(\theta_1), \ldots, \boldsymbol{a}_{\mathrm{t}}(\theta_S)], \qquad \boldsymbol{\Psi} = \begin{bmatrix} \psi_1 & & \\ & \ddots & \\ & & \psi_S \end{bmatrix}, \qquad \psi_i = e^{j2\pi\Delta\sin\theta_i}$$

ESPRIT

- Stack $\boldsymbol{X}_{\mathrm{t}}$ and $\boldsymbol{X}_{\mathrm{b}}$ in a matrix $\boldsymbol{Y}$ and compute the economy size SVD

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{X}_{\mathrm{t}} \\ \boldsymbol{X}_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{\mathrm{t}} \\ \boldsymbol{A}_{\mathrm{t}}\boldsymbol{\Psi} \end{bmatrix} \boldsymbol{B}\boldsymbol{S} = \boldsymbol{U}_y \boldsymbol{\Sigma}_y \boldsymbol{V}_y^H$$

- The main goal of this SVD is to realize a compression of the columns

$$\boldsymbol{U}_y = \begin{bmatrix} \boldsymbol{U}_{\mathrm{t}} \\ \boldsymbol{U}_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{\mathrm{t}} \\ \boldsymbol{A}_{\mathrm{t}}\boldsymbol{\Psi} \end{bmatrix} \boldsymbol{T},$$

$\boldsymbol{T}$ is an $S \times S$ invertible matrix

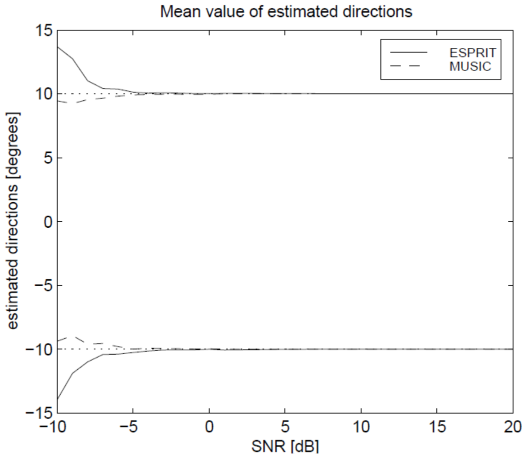
- Using these expression we can derive

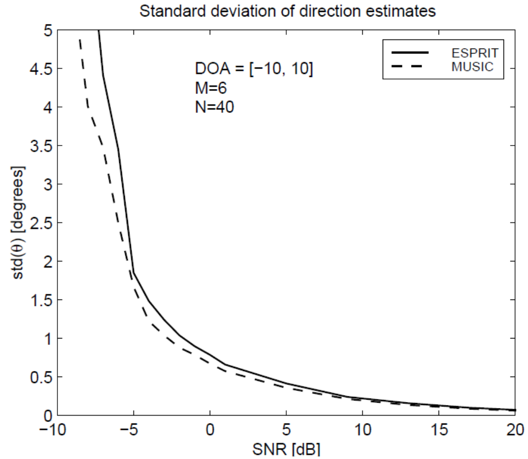
$$\boldsymbol{U}_{\mathrm{t}}^{\dagger} = \boldsymbol{T}^{-1}\boldsymbol{A}_{\mathrm{t}}^{\dagger} \quad \Rightarrow \quad \boxed{\boldsymbol{U}_{\mathrm{t}}^{\dagger}\boldsymbol{U}_{\mathrm{b}} = \boldsymbol{T}^{-1}\boldsymbol{\Psi}\boldsymbol{T}}$$

- Thus $\boldsymbol{T}^{-1}$ and $\boldsymbol{\Psi}$ are given by the eigenvectors and eigenvalues of $\boldsymbol{U}_{\mathrm{t}}^{\dagger}\boldsymbol{U}_{\mathrm{b}}$
- From $\boldsymbol{\Psi}$ we can derive $\{\psi_i\}_{i=1}^{S}$ and hence $\{\theta_i\}_{i=1}^{S}$

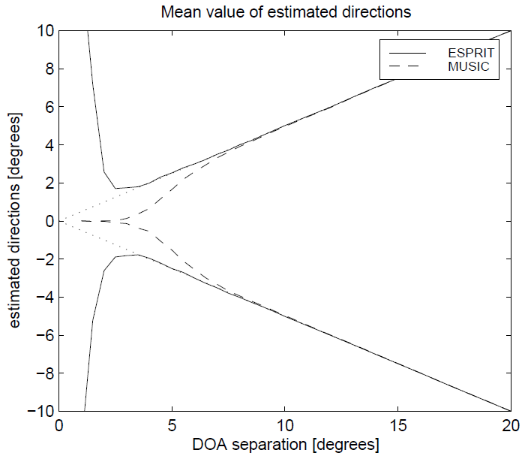
ESPRIT versus MUSIC

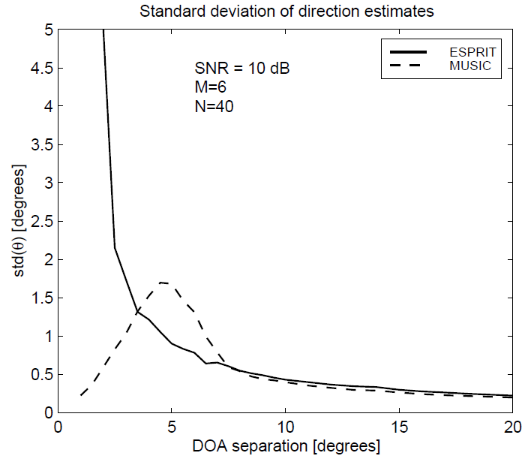
- Varying SNR





- Varying separation





Sparse Reconstruction [Malioutov et al, 2005]

- The sparse reconstruction idea is based on rewriting the data model using a grid of $\tilde{S}$ angles $\{\tilde{\theta}_j\}_{j=1}^{\tilde{S}}$ spanning the angular space of interest

$$\boldsymbol{x}[k] = \sum_{i=1}^{S} \boldsymbol{a}(\theta_i)\beta_i s_i[k] = \sum_{j=1}^{\tilde{S}} \boldsymbol{a}(\tilde{\theta}_j)\tilde{s}_j[k] = \underbrace{[\boldsymbol{a}(\tilde{\theta}_1), \ldots, \boldsymbol{a}(\tilde{\theta}_{\tilde{S}})]}_{\tilde{\boldsymbol{A}}} \underbrace{\begin{bmatrix} \tilde{s}_1[k] \\ \vdots \\ \tilde{s}_{\tilde{S}}[k] \end{bmatrix}}_{\tilde{\boldsymbol{s}}[k]}$$

$$\boxed{\boldsymbol{x}[k] = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{s}}[k]}$$

  - Since this grid of angles is known $\Rightarrow$ the system matrix $\tilde{\boldsymbol{A}}$ is known
  - If $\tilde{\theta}_j = \theta_i$ then $\tilde{s}_j[k] = \beta_i s_i[k]$, otherwise $\tilde{s}_j[k] = 0$ $\Rightarrow$ $\tilde{\boldsymbol{s}}[k]$ is S-sparse
- The goal is to solve $\boldsymbol{x}[k] = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{s}}[k]$ for $\tilde{\boldsymbol{s}}[k]$ (non-zero entries reveal $\{\theta_i\}_{i=1}^{S}$)
  - However, $\tilde{\boldsymbol{s}}[k]$ cannot be solved using least squares because $M \ll \tilde{S}$
  - Hence, additional constraints are needed such as sparsity

Sparse Reconstruction

- Using a sparsity constraint the problem becomes

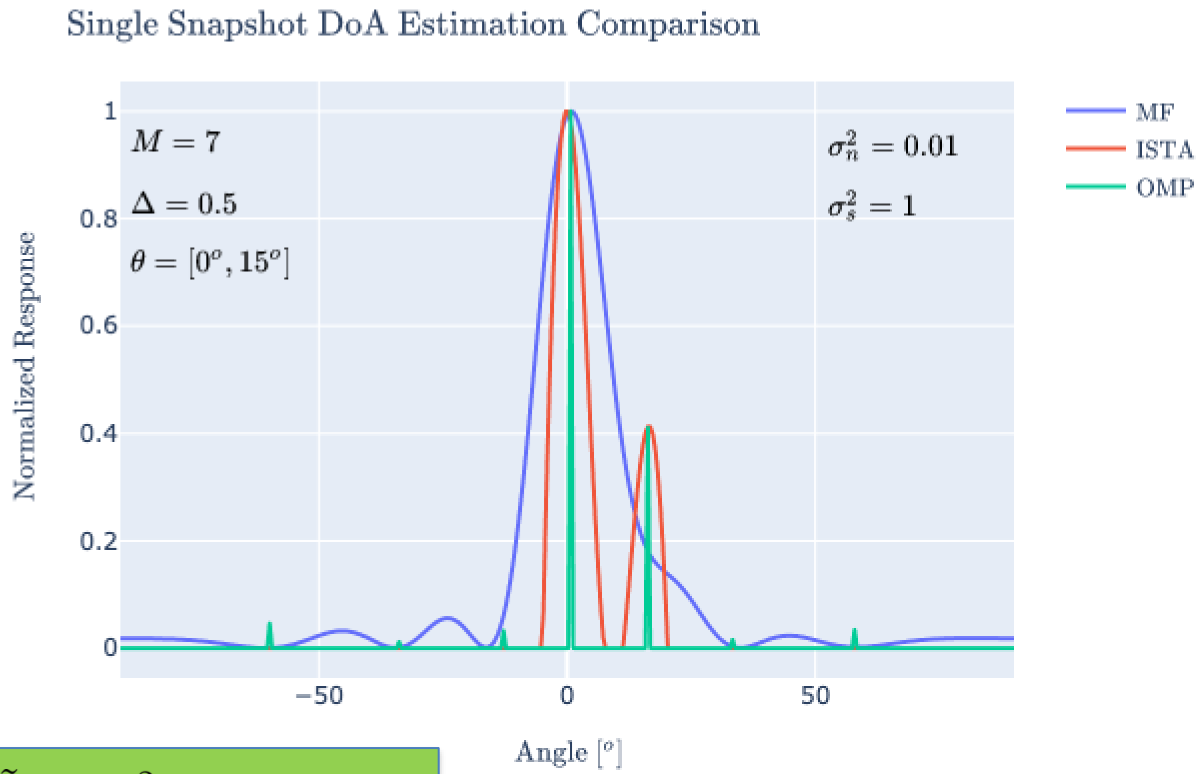
$$\min_{\tilde{\boldsymbol{s}}[k]} ||\boldsymbol{x}[k] - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{s}}[k]||^2 \quad \text{s.t.} \quad ||\tilde{\boldsymbol{s}}[k]||_0 = S$$

- This problem is NP hard but can be relaxed to a convex problem as

$$\boxed{\min_{\tilde{\boldsymbol{s}}[k]} ||\boldsymbol{x}[k] - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{s}}[k]||^2 + \lambda||\tilde{\boldsymbol{s}}[k]||_1} \qquad ||\tilde{\boldsymbol{s}}[k]||_1 = \sum_{i=1}^{\tilde{S}} |\tilde{s}_i[k]|$$

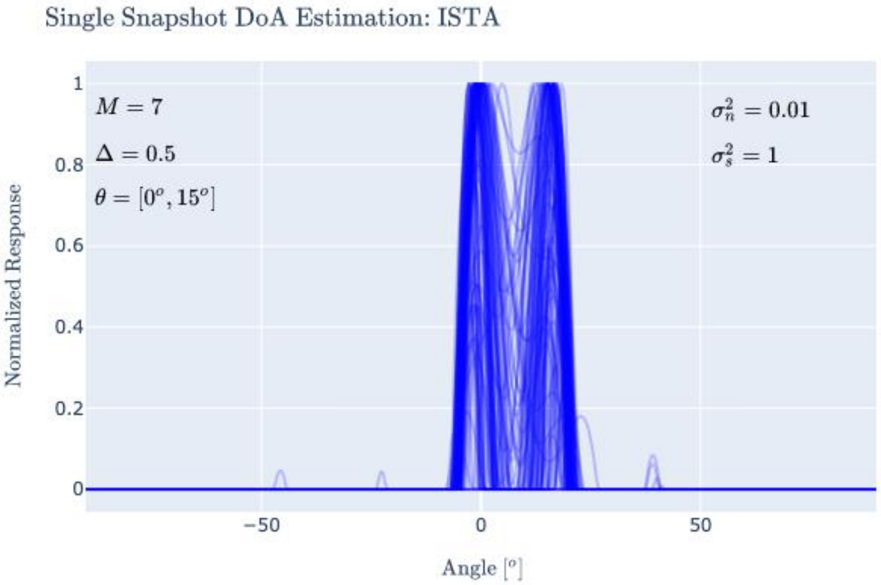
- The theory of compressive sensing has shown that the two above problems can have the same solution under some conditions related to the structure of $\tilde{\boldsymbol{A}}$
- Typical algorithms that can be employed are
  - Iterative soft tresholding algorithm (ISTA) → [Daubechies et al, 2004] [Beck-Teboulle, 2009]
  - Orthogonal matching pursuit (OMP) → [Davis et al, 1997] [Tropp, 2004] [Donoho et al, 2006]

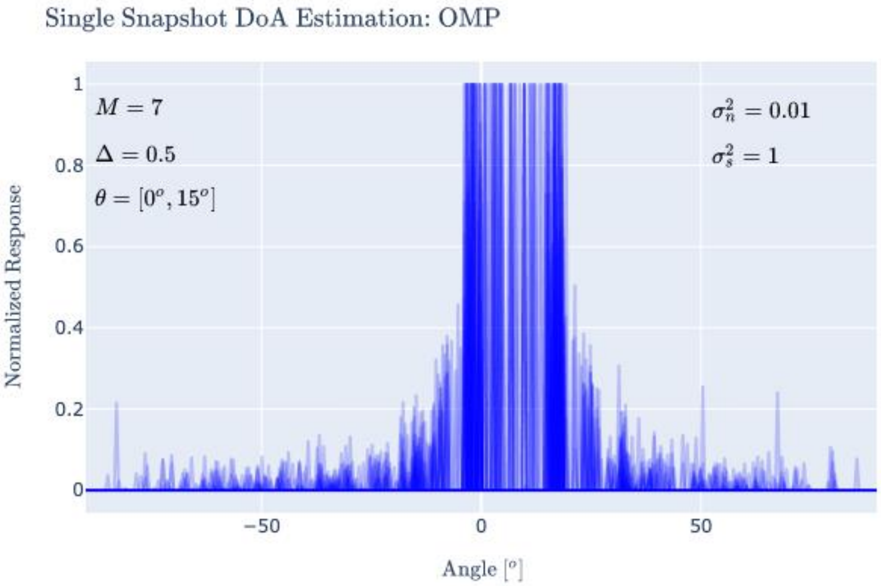
Sparse Reconstruction



$$\min_{\tilde{\boldsymbol{s}}[k]} \|\boldsymbol{x}[k] - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{s}}[k]\|^2 + \lambda\|\tilde{\boldsymbol{s}}[k]\|_1$$

Sparse Reconstruction





Result of multiple realizations

Sparse Reconstruction

- We can extend this framework to multiple snapshots by defining

$$\boldsymbol{X} = [\boldsymbol{x}[1], \ldots, \boldsymbol{x}[K]], \qquad \tilde{\boldsymbol{S}} = [\tilde{\boldsymbol{s}}[1], \ldots, \tilde{\boldsymbol{s}}[K]]$$

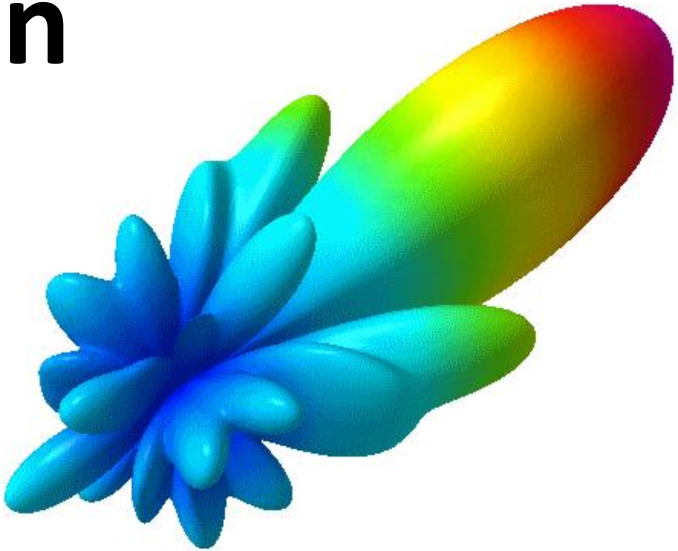
- The optimization problem then becomes

$$\boxed{\min_{\tilde{\boldsymbol{S}}} \|\boldsymbol{X} - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{S}}\|^2 + \lambda\|\tilde{\boldsymbol{S}}\|_{2,1}} \qquad \|\tilde{\boldsymbol{S}}\|_{2,1} = \sum_{i=1}^{\tilde{S}} \sqrt{\sum_{k=1}^{K} |\tilde{s}_i[k]|^2}$$

- Similar algorithms as in the single snapshot case can be employed
- The problem can be simplified by using the l1-SVD idea [Malioutov et al, 2005]
  - Use the SVD to reduce the rank of $\boldsymbol{X}$ to S : $\boldsymbol{X} \approx \underset{M \times S}{\boldsymbol{U}_x}\,\underset{S \times S}{\boldsymbol{\Sigma}_x}\,\underset{S \times K}{\boldsymbol{V}_x^H}$
  - Then we solve the reduced problem

$$\boxed{\min_{\tilde{\boldsymbol{S}}} \|\boldsymbol{U}_x\boldsymbol{\Sigma}_x - \tilde{\boldsymbol{A}}\tilde{\boldsymbol{S}}\|^2 + \lambda\|\tilde{\boldsymbol{S}}\|_{2,1}}$$

Part III: Advanced Methods and Array Design



Advanced Methods and Array Design

- Compressive array
  - Compressive sensing (CS)
  - CS in spatial domain
- Covariance based processing
  - Compressive covariance sensing (CCS)
  - CCS in spatial domain
  - CCS based array design
  - Virtual array principle
- Performance based array design
  - Sparse sensing
  - Convex and submodular optimization

Advanced Methods and Array Design

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- Covariance based processing
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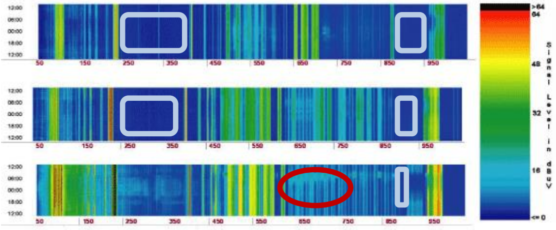
  - CCS based array design
  - Virtual array principle
- Performance based array design
  - Sparse sensing
  - Convex and submodular optimization

Compressive Sensing

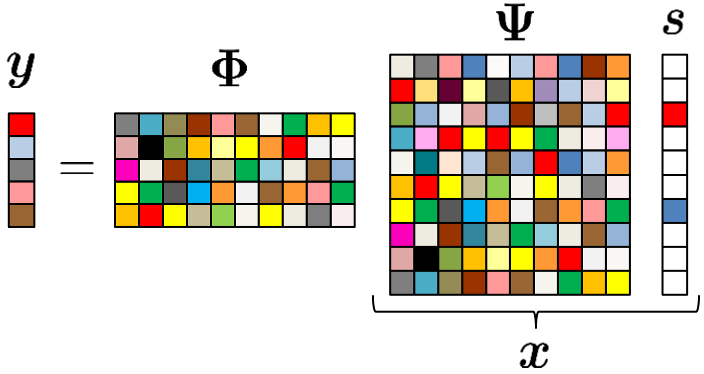
- Due to the required high sampling rates, compression is useful

cognitive radio



massive MIMO

- Compression after acquisistion does not simplify sensing
- A popular alternative is compressive sensing (CS) = joint acquisition and sensing

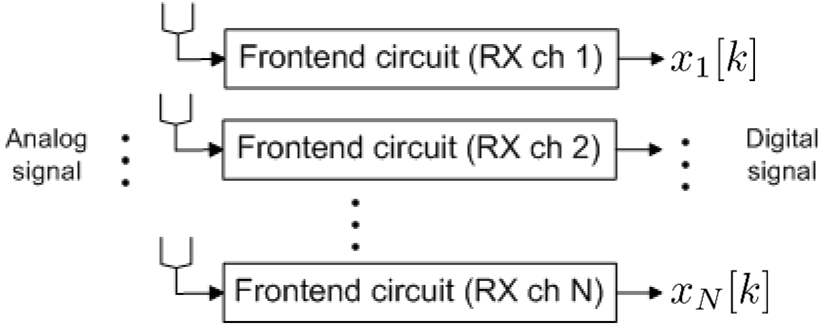


[Tropp, 2004] [Donoho, 2006] [Candès et al, 2006]

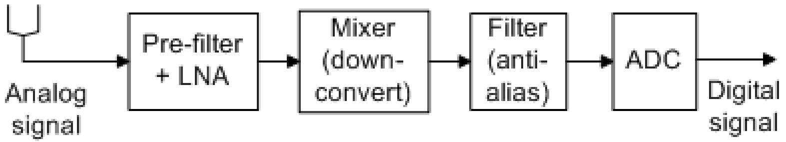
  – Random linear projections of Nyquist rate sampled signal
  – Multiple sparse reconstruction techniques exist to solve this

CS in Spatial Domain

RX array, *N* antennas



each RX chain:



high angular resolution

large aperture

many antennas

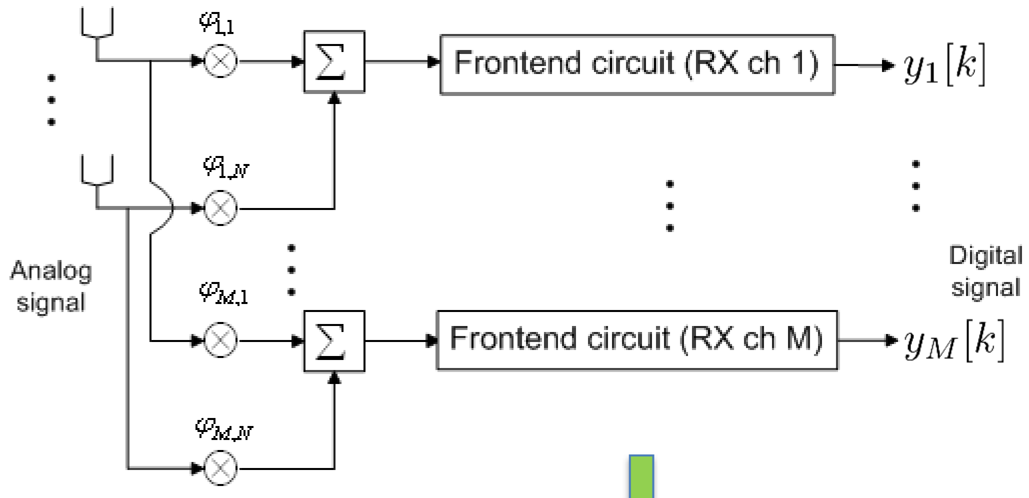
many RX chains

high power consumption

Goal: Decrease the number of RX chains without loosing too much performance

CS in Spatial Domain

Solution: Compressive array → $\boldsymbol{y}[k] = \boldsymbol{\Phi}\boldsymbol{x}[k]$



compression

from N to $M<N$ RX chains

Compression matrix $\boldsymbol{\Phi}$:

- Dense matrix: analog beamforming, beamspace
  [Wang-Leus-Pandharipande, 2009]
  [Wang-Leus, 2010]
  [Venkateswaran-van der Veen, 2010]
- Sparse matrix: subarrays
  [Moffet, 1968] [Hoctor-Kassam, 1990]

DoA Estimation

$$\boldsymbol{x}[k] = \boldsymbol{A}\boldsymbol{s}[k] + \boldsymbol{n}_t \quad \Rightarrow \quad \boldsymbol{R}_x = \boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H + \boldsymbol{R}_n$$

$$\boldsymbol{y}[k] = \boldsymbol{\Phi}\boldsymbol{x}[k]$$

compression

$$\boldsymbol{y}[k] = \boldsymbol{B}\boldsymbol{s}[k] + \boldsymbol{m}[k] \quad \Rightarrow \quad \boldsymbol{R}_y = \boldsymbol{B}\boldsymbol{R}_s\boldsymbol{B}^H + \boldsymbol{R}_m$$

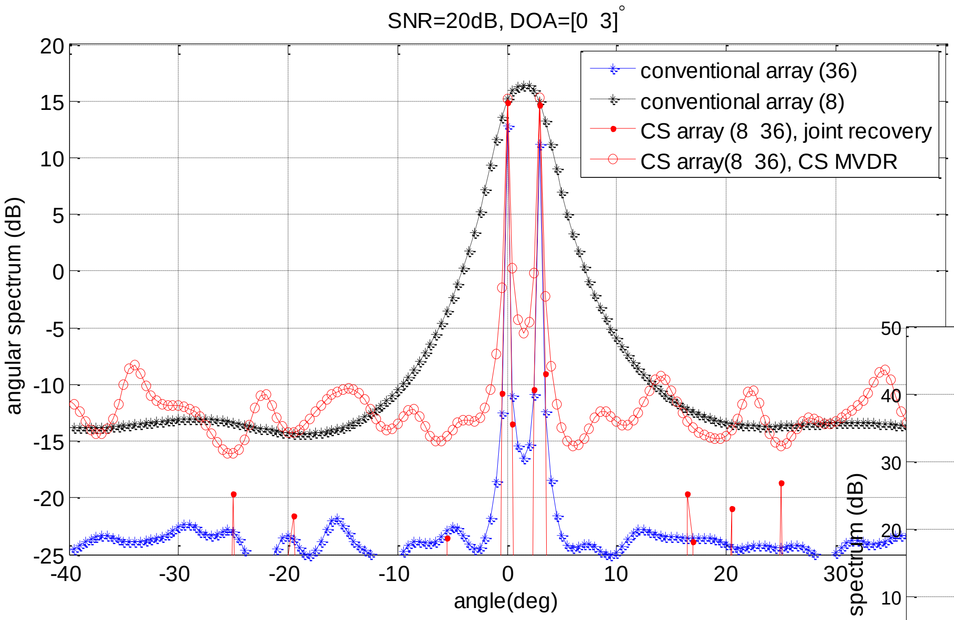
$$\boldsymbol{B} = \boldsymbol{\Phi}\boldsymbol{A} = [\boldsymbol{b}(\theta_1), \ldots, \boldsymbol{b}(\theta_S)], \qquad \boldsymbol{m}[k] = \boldsymbol{\Phi}\boldsymbol{n}[k]$$

- Based on $\boldsymbol{R}_y$ and $\boldsymbol{b}(\theta)$ traditional DoA methods can be used (see earlier)
  - Beamforming-based methods
  - MUSIC
  - Sparse reconstruction
  - ESPRIT is not possible since we don't have a ULA
- They require (only work well for) more outputs than sources, $M > S$

Simulations Results

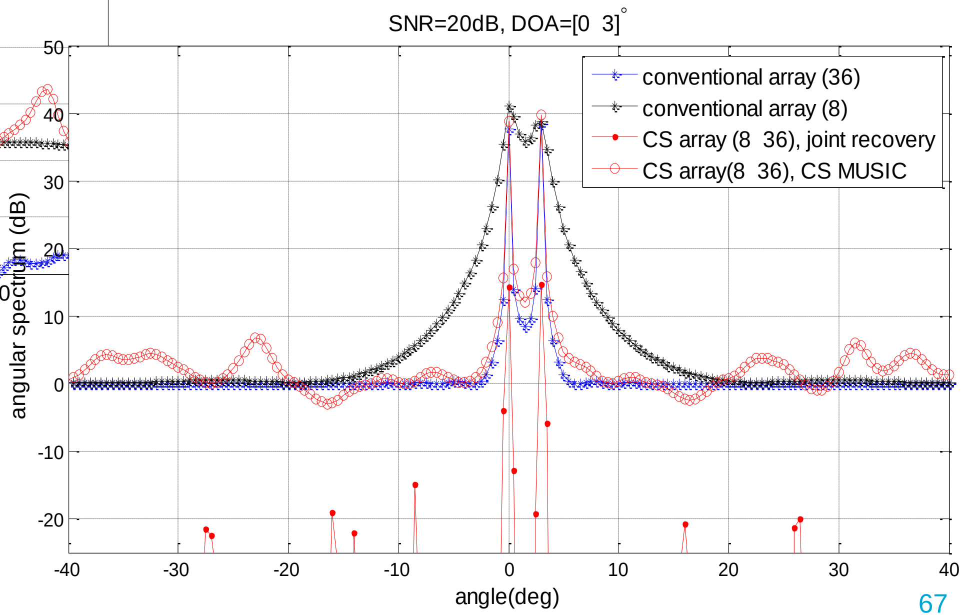
- ULA (uniform linear array), $\Delta = 1/2$
- $S = 2$ sources, uncorrelated, BPSK, DoAs $\theta_1 = 0°, \ \theta_2 = 30°$
- Conventional array: $N = 36$ and $N = 8$ antenna elements
- Compressive array: from $N = 36$ we compress to $M = 8$ RX chains
- Scanning resolution: $\tilde{S} = 360$
- Compression matrix:
  - random Gaussian matrix: entries zero mean and variance $1/M$
  - random selection matrix: randomly selecting M from N elements
- Sparse reconstruction: M-FOCUSS

Random Gaussian

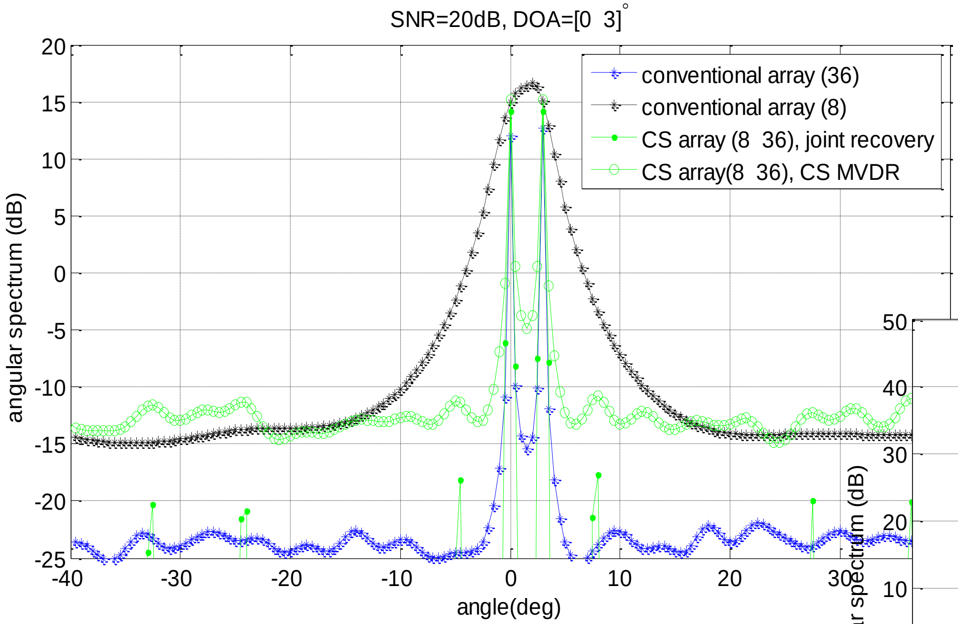


MVDR

MUSIC

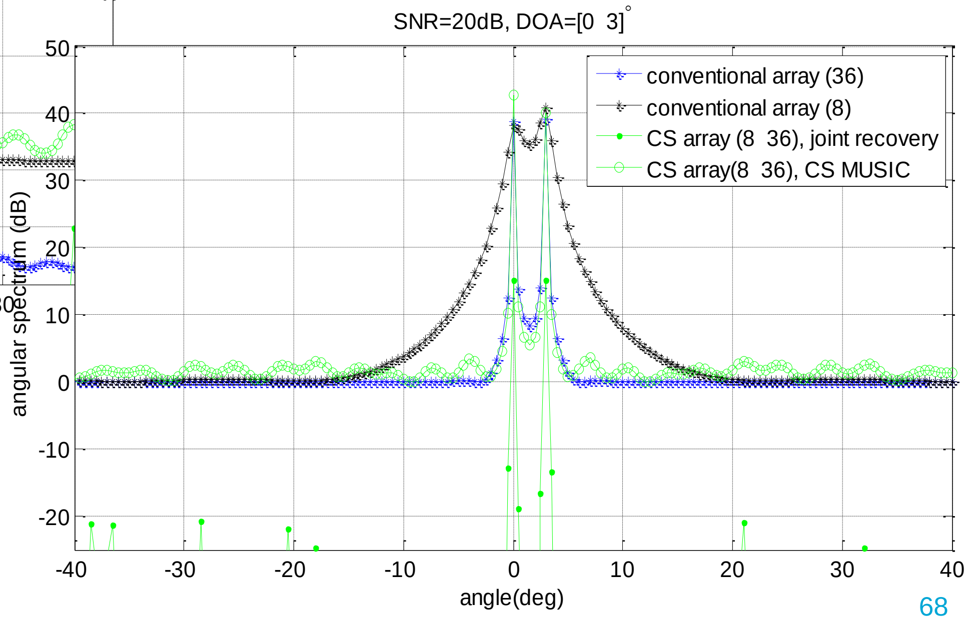


Random Selection



MVDR

MUSIC



Advanced Methods and Array Design

- Compressive array
  - Compressive sensing (CS)
  - CS in spatial domain
- Covariance based processing
  - Compressive covariance sensing (CCS)
  - CCS in spatial domain
  - CCS based array design
  - Virtual array principle
- Performance based array design
  - Sparse sensing
  - Convex and submodular optimization

Compressive Covariance Sensing

- CS methods estimate the signal itself (or its spectrum)
  - Leads to an underdetermined problem that requires a sparsity constraint
  - High computational complexity
  - Difficult performance analysis

Observation:

- Many applications just require second-order statistics (or the power spectrum)
  - Cognitive radio: temporal power spectrum
  - Radio astronomy: spatial power spectrum
- This paves the way for methods that are less complex and easier to analyze

→ Compressive covariance sensing (CCS)

CCS in Spatial Domain

$\boldsymbol{x}_t \in \mathbb{C}^N \rightarrow \boxed{\text{compression } \boldsymbol{\Phi}} \rightarrow \boldsymbol{y}_t \in \mathbb{C}^M$

uncompressed signal — compressed signal

$\boldsymbol{R}_{\boldsymbol{x}} = \mathrm{E}\{\boldsymbol{x}_t \boldsymbol{x}_t^H\}$ $\qquad$ $\boldsymbol{R}_{\boldsymbol{y}} = \mathrm{E}\{\boldsymbol{y}_t \boldsymbol{y}_t^H\}$

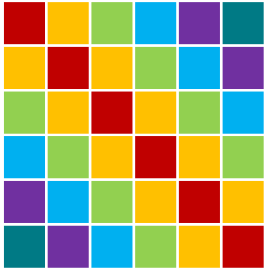
- Problem statement:

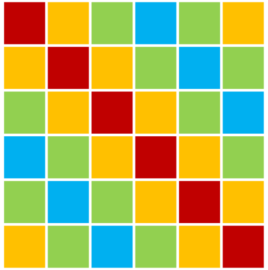
> Estimate $\boldsymbol{R}_{\boldsymbol{x}}$ from $\{\boldsymbol{y}_t\}_t$ or $\hat{\boldsymbol{R}}_{\boldsymbol{y}}$ exploiting structure in $\boldsymbol{R}_{\boldsymbol{x}}$

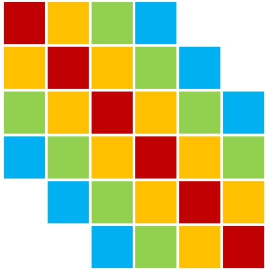
- Once $\boldsymbol{R}_{\boldsymbol{x}}$ is estimated we can use any ULA-based DoA estimation technique
  - Beamforming based methods
  - MUSIC
  - Sparse reconstruction
  - ESPRIT becomes also possible now because of the ULA property

Covariance Structure

- All covariance matrices are Hermitian and positive semi-definite
- Typical structures for $\boldsymbol{R_x}$
  - Toeplitz
    - Stationarity over space
    - Sum of exponentials
  - Circulant
    - Rows/colums are circularly shifted
    - Sum of exponentials on uniform grid
  - d-banded
    - Toeplitz with info only in d subdiagonals
    - Spatial MA/AR processes (not related to our model)



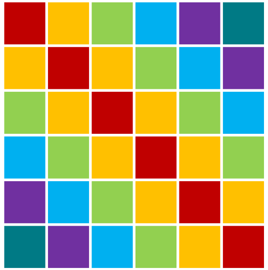




Covariance Structure

- Encompassing model: basis expansion model (BEM)

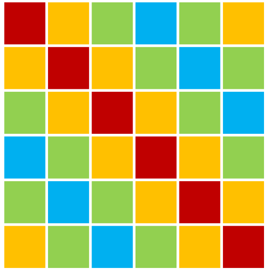
$$\mathcal{S} = \left\{ \boldsymbol{R}_{\boldsymbol{x}} = \sum_{i=1}^{S} \alpha_i \boldsymbol{R}_i, \quad \alpha_i \in \mathbb{R} \right\}$$



Toeplitz

$S = 2N - 1$

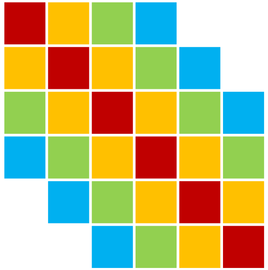
real unknowns



circulant

$S = N$

real unknowns



d-banded

$S = 2d + 1$

real unknowns

Covariance Estimation

[Leus-Ariananda, 2011]
[Ariananda-Leus, 2012]

- We vectorize the involved covariance matrices

$$\boldsymbol{r_x} = \text{vec}\,\boldsymbol{R_x}, \qquad \boldsymbol{r_y} = \text{vec}\,\boldsymbol{R_y}, \qquad \boldsymbol{r}_i = \text{vec}\,\boldsymbol{R}_i$$

- We can then establish a relation between $\boldsymbol{r_x}$ and $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_S]^T$

$$\boldsymbol{R_x} = \textstyle\sum_{i=1}^{S} \alpha_i \boldsymbol{R}_i \qquad \Rightarrow \qquad \boldsymbol{r_x} = \boldsymbol{T}\boldsymbol{\alpha}, \quad \boldsymbol{T} = [\boldsymbol{r}_1, \dots, \boldsymbol{r}_S]$$

- The same can be done for $\boldsymbol{r_x}$ and $\boldsymbol{r_y}$

$$\boldsymbol{R_y} = \boldsymbol{\Phi}\boldsymbol{R_x}\boldsymbol{\Phi}^H \qquad \Rightarrow \qquad \boldsymbol{r_y} = (\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{r_x}$$

- Based on an estimated $\boldsymbol{r_y}$ we can recover $\boldsymbol{\alpha}$ using least squares

$$\boldsymbol{r_y} = (\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T}\boldsymbol{\alpha} \qquad \Rightarrow \qquad \hat{\boldsymbol{\alpha}} = [(\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T}]^\dagger \hat{\boldsymbol{r}}_{\boldsymbol{y}}$$

- Finally we can obtain an estimate for $\boldsymbol{r_x}$ as

$$\boxed{\hat{\boldsymbol{r}}_{\boldsymbol{x}} = \boldsymbol{T}[(\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T}]^\dagger \hat{\boldsymbol{r}}_{\boldsymbol{y}}}$$

Covariance Estimation

$$\boldsymbol{r_y} = \underbrace{\underbrace{(\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})}_{M^2 \times N^2}\boldsymbol{T}}_{M^2 \times S}\boldsymbol{\alpha}$$

$$\boxed{\hat{\boldsymbol{\alpha}} = [(\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T}]^{\dagger}\hat{\boldsymbol{r}}_{\boldsymbol{y}}}$$

- When $M^2 > S$ we have an overdetermined system
- This can happen even under compression, $M < N$
- Thus also under less outputs than sources, $M < S$
- Unique reconstruction if $(\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T}$ full column rank

$\Rightarrow$ design of $\boldsymbol{\Phi}$ is critical!

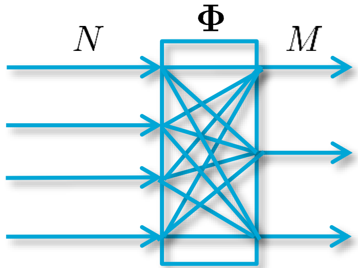
- Estimation can include additional constraints

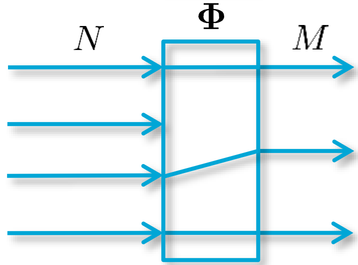
$$\min_{\boldsymbol{\alpha}} \|\hat{\boldsymbol{r}}_{\boldsymbol{y}} - (\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T}\boldsymbol{\alpha}\|^2$$

$$\text{s.t.} \begin{cases} \boldsymbol{FT\alpha} \succeq \boldsymbol{0} & \text{PSD non-negative} \\ \|\boldsymbol{FT\alpha}\|_0 \leq \mu & \text{PSD sparse} \\ \boldsymbol{R_x} \succeq \boldsymbol{0} & \text{Covariance PSD} \end{cases}$$

CCS Based Array Design

- Compressive array
  - Dense matrix $\boldsymbol{\Phi}$
  - Sparse matrix $\boldsymbol{\Phi}$





- Goals of sampler design:
  - Conditions on $\boldsymbol{\Phi}$ to allow for estimation of $\boldsymbol{R_x}$
  - Maximize the compression ratio $\rho = N/M$
- An admissable covariance sampler allows the recovery of $\boldsymbol{\alpha}$ from $\boldsymbol{R_y}$

$$\boldsymbol{r_y} = (\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T\alpha}$$

➡ The matrix $(\boldsymbol{\Phi}^* \otimes \boldsymbol{\Phi})\boldsymbol{T}$ has full column rank

$$\boldsymbol{R_x} = \sum_{i=1}^{S} \alpha_i \boldsymbol{R}_i \quad \Rightarrow \quad \boldsymbol{R_y} = \sum_{i=1}^{S} \alpha_i \boldsymbol{\Phi} \boldsymbol{R}_i \boldsymbol{\Phi}^H$$

➡ The linear indepence in $\{\boldsymbol{R}_i\}_{i=1}^{S}$ is preserved in $\{\boldsymbol{\Phi}\boldsymbol{R}_i\boldsymbol{\Phi}^H\}_{i=1}^{S}$

Sparse Compression

- In sparse array design, the matrix $\mathbf{\Phi}$ selects a subset of antennas $\mathcal{I}$

$$\boxed{\mathcal{I} \subset \{0, \ldots, N-1\}}$$

- Toeplitz subspace: $\mathbf{\Phi}$ covariance sampler $\Leftrightarrow$ $\mathcal{I}$ sparse ruler
  - Optimal sparse array: $\mathcal{I}$ minimal sparse ruler

    [Rédei-Rényi, 1949] [Leech, 1956] [Pearson et al, 1990] [Romero-LópezValcarce-Leus, 2015]
- Circulant subspace: $\mathbf{\Phi}$ covariance sampler $\Leftrightarrow$ $\mathcal{I}$ circular sparse ruler
  - Optimal sparse array: $\mathcal{I}$ minimal circular sparse ruler

    [Romero-LópezValcarce-Leus, 2015]
- d-banded subspace: $\mathbf{\Phi}$ covariance sampler $\Leftrightarrow$ $\mathcal{I}$ incomplete sparse ruler

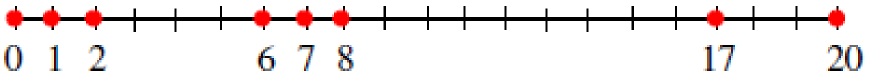
  [Ariananda-Leus, 2012] [Romero-LópezValcarce-Leus, 2015]

Sparse Rulers

- Difference set: $\Delta(\mathcal{I}) = \{|i_1 - i_2|,\ \forall i_1, i_2 \in \mathcal{I}\}$
- Sparse ruler:

$$\boxed{\mathcal{I} \text{ is a length-}(N-1) \text{ sparse ruler} \Longleftrightarrow \Delta(\mathcal{I}) = \{0, \ldots, N-1\}}$$

$N = 21$:



$$\frac{N}{\lceil\sqrt{3(N-1)}\rceil} \leq \rho_{\max} \leq \frac{N}{\sqrt{2.435(N-1)}}$$

- Minimal sparse ruler

  [Rédei-Rényi, 1949] [Leech, 1956] [Wichmann, 1963] [Moffet, 1968] [Miller, 1971] [Wild, 1987] [Pearson et al, 1990] [Linebarger et al, 1993] [Ariananda-Leus, 2012]

- Suboptimal designs: nested, co-prime

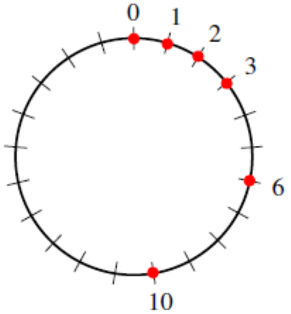
  [Wichmann, 1963] [Pearson et al, 1990] [Linebarger et al, 1993] [Pumphrey, 1993] [Pal-Vaidyanathan, 2010] [Pal-Vaidyanathan, 2011]

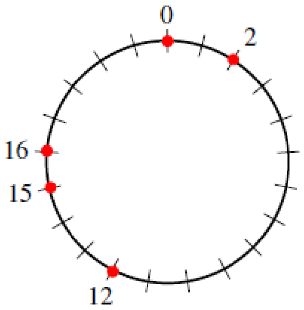
Circular Sparse Rulers

- Modular difference set: $\Delta_N(\mathcal{I}) = \{(i_1 - i_2) \mod N, \ \forall i_1, i_2 \in \mathcal{I}\}$
- Circular sparse ruler:

$$\mathcal{I} \text{ is a length-}(N-1) \text{ circular sparse ruler} \Longleftrightarrow \Delta_N(\mathcal{I}) = \{0, \ldots, N-1\}$$

$N = 21$:





- Minimal circular sparse ruler

$$\frac{N}{\lceil \sqrt{3\lfloor N/2 \rfloor} \rceil} \leq \rho_{\max} \leq \frac{2N}{2 + \sqrt{4N-3}}$$

[Singer, 1938] [Miller, 1971] [Ariananda-Leus, 2012] [Romero-Leus, 2013] [Krieger-Kochman-Wornell, 2013] [Romero-LópezValcarce-Leus, 2015]

Dense Compression

- Design: Similar to CS we use random designs

$\mathbf{\Phi}$ covariance sampler ⟺ $\mathbf{\Phi}$ drawn from continuous distribution and $M^2 \geq S$

– Toeplitz subspace: $\rho_{\max} \approx \sqrt{\frac{N^2}{2N-1}}$

– Circulant subspace: $\rho_{\max} \approx \sqrt{N}$

– d-banded subspace: $\rho_{\max} \approx \sqrt{\frac{N^2}{2d-1}}$

[Romero-LópezValcarce-Leus, 2015]

Virtual Array Principle

- Exploiting uncorrelated sources, i.e., $\boldsymbol{R_s}$ is diagonal, we can obtain

[Pillai et al, 1985] [Abramovich et al, 1998, 1999]
[Pal-Vaidyanathan, 2010, 2011] [Shakeri-Ariananda-Leus, 2012]
[Yen-Tsai-Wang, 2013] [Krieger-Kochman-Wornell, 2013]

$$\boldsymbol{r_y} = \mathrm{vec}\, \boldsymbol{R_y} = \mathrm{vec}(\overset{\overset{M \times S}{\uparrow}}{\boldsymbol{B}} \boldsymbol{R_s} \boldsymbol{B}^H) = (\overset{\overset{M^2 \times S}{\uparrow}}{\boldsymbol{B}^* \odot \boldsymbol{B}})\, \mathrm{diag}\, \{\boldsymbol{R_s}\}$$

- $\boldsymbol{B}^* \odot \boldsymbol{B}$ represents a virtual array: M^2observations versus S sources
- Problem: virtual sources are constant or fully coherent
  - Gridding and sparse recovery
  - Smoothing by taking subarrays

Virtual Array Principle

- Gridding [Shakeri-Ariananda-Leus, 2012]

$$\boxed{\boldsymbol{r}_{\boldsymbol{y}} = \operatorname{vec} \boldsymbol{R}_{\boldsymbol{y}} = \operatorname{vec}(\tilde{\boldsymbol{B}} \boldsymbol{R}_{\tilde{\boldsymbol{s}}} \tilde{\boldsymbol{B}}^H) = \underbrace{(\tilde{\boldsymbol{B}}^* \odot \tilde{\boldsymbol{B}})}_{M^2 \times \tilde{S}} \operatorname{diag}\{\boldsymbol{R}_{\tilde{\boldsymbol{s}}}\}}$$

$M^2 \geq \tilde{S}$: least squares $\quad$ $M^2 < \tilde{S}$: add sparsity/positivity

- Smoothing by stacking subarrays followed by standard methods
  - Virtual array should be uniform (leading to a Vandermonde matrix)

    ⇒ Only antenna selection is possible

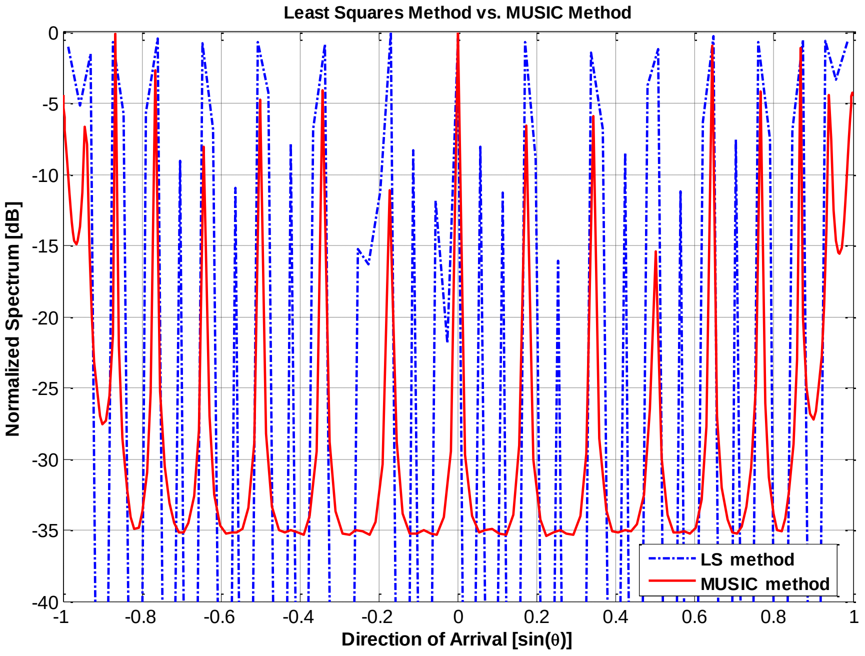
    ⇒ Sparse ruler based antenna selection

Simulations

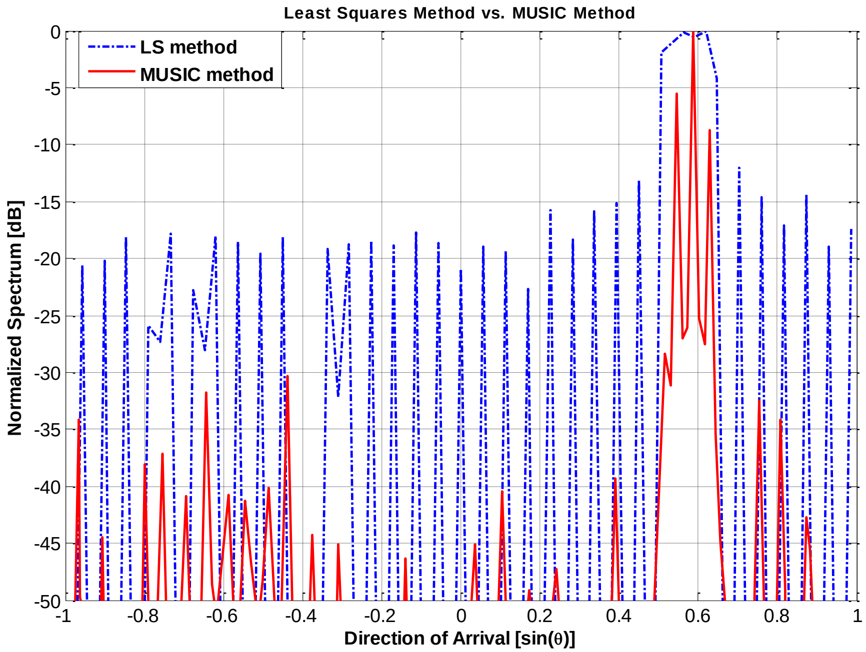
- Least squares and MUSIC reconstruction
- We consider the space of Toeplitz matrices
- ULA of $N = 36$ available antenna positions
  - Minimal sparse ruler array $M = 10$ → Virtual ULA of 2x36-1antennas
  - Two-level nested array $M = 11$
    - Inner array of 5 and outer array of 6 antennas
    - Virtual ULA of 2x36-1 antennas
  - Co-prime array
    - 9 antennas spacing 2 and 3 antennas spacing 9
    - Virtual ULA of only 2x20-1 antennas
- 1600 time samples
- SNR of 0 dB

Simulations

- Minimal sparse ruler array



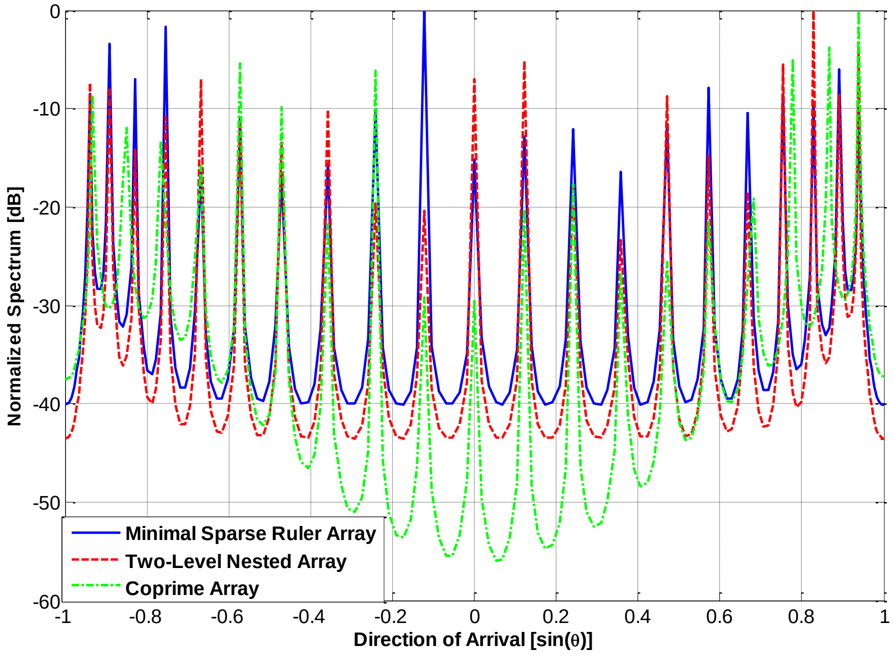
17 sources with 10 degrees of separation



continuous source from 30 to 40 degrees

Simulations

- Comparison of different array structures



21 sources with 7 degrees of separation

Advanced Methods and Array Design

- Compressive array
  - Compressive sensing (CS)
  - CS in spatial domain
- Covariance based processing
  - Compressive covariance sensing (CCS)
  - CCS in spatial domain
  - CCS based array design
  - Virtual array principle
- Performance based array design
  - Sparse sensing
  - Convex and submodular optimization

Performance Based Array Design

- From a full array of N antennas (e.g., Nyquist sampled array)
  - Design the optimal subarray of $M < N$ antennas
  - Define an appropriate optimality criterion

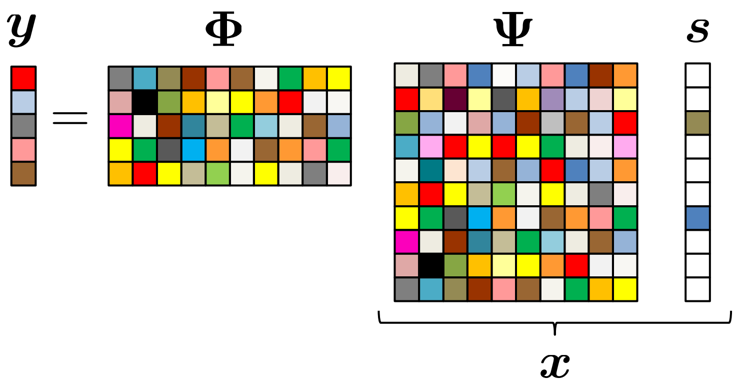
→ **sparse sensing/sampling, sensor/antenna selection/placement**

[Blu et al, 2008] [Vaidyanathan-Pal, 2011]

- Why sparse sensing?
  - Economical constraints
  - Limited physical space
  - Limited storage space
  - Reduce communications/processing overhead

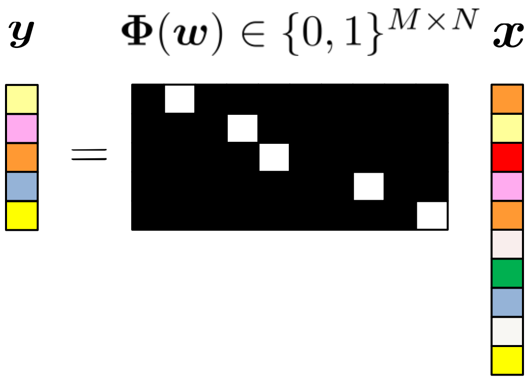
Sparse Sensing

Compressive sensing



- Sparse signal needed
- Random, dense sampler
- Robust compression
- Task is sparse signal reconstruction

Sparse sensing



- Signal does not need to be sparse
- Deterministic, sparse sampler
- Practical, controlable compression
- Any desired inference task

Design Problem

Select the "best" subarray of antennas out of the candidate antennas that guarantee a certain desired statistical inference performance

$$\arg\min_{\boldsymbol{w}} \|\boldsymbol{w}\|_0 \quad \text{s.t.} \quad f(\boldsymbol{w}) \leq \lambda, \quad \boldsymbol{w} \in \{0, 1, \}^N \quad \Leftrightarrow \quad \arg\min_{\boldsymbol{w}} f(\boldsymbol{w}) \quad \text{s.t.} \quad \|\boldsymbol{w}\|_0 = M, \quad \boldsymbol{w} \in \{0, 1, \}^N$$

$f(\boldsymbol{w})$: inference metric

λ : prescribed performance

N : # candidate samples

M : # selected samples

- This is a nonconvex Boolean problem
- Exact solutions: exhaustive search or branch-and-bound methods [Lawler-Wood, 1966]
- Suboptimal solutions:
  - Convex optimization (polynomial time)
  - Submodular optimization (linear time)

Suboptimal Solutions

- Convex optimization [Joshi-Boyd, 2009] [Chepuri-Leus, 2015]
  - Convex relaxation for $\{0, 1\}$, $f(\boldsymbol{w})$
  - Thresholding or randomization to obtain a Boolean solution
  - Typically a semidefinite program
- Submodular optimization for maximization [Krause et al, 2008] [Ranieri et al, 2014]
  - A greedy search is performed
  - For a submodular monotone function

$$\forall \mathcal{Y} \subseteq \mathcal{X} \subset \{1, \ldots, N\}, s \in \{1, \ldots, N\} \backslash \mathcal{X}$$
$$f(\mathcal{Y} \cup s) - f(\mathcal{Y}) \geq f(\mathcal{X} \cup s) - f(\mathcal{X}) \quad \text{and} \quad f(\mathcal{X} \cup s) \geq f(\mathcal{X})$$

the greedy solution is $1 - 1/e$ optimal [Nemhauser et al, 1978]

Application to Estimation

[Chepuri-Leus, 2015] [Liu et al, 2016]

- Suppose $\boldsymbol{x}$ depends on an unknown parameter vector $\boldsymbol{\theta}$ (e.g., DoAs)
- Then the selection can be written as

$$\boldsymbol{y} = \mathrm{diag}(\boldsymbol{w})\boldsymbol{x}, \quad \boldsymbol{w} = [w_1, w_2, \ldots, w_N]^T, w_n \in \{0, 1\}$$

- This means the original pdf $p(\boldsymbol{x}; \boldsymbol{\theta})$ becomes $p(\boldsymbol{y}; \boldsymbol{w}, \boldsymbol{\theta})$ after selection
- We could then optimize $\boldsymbol{w}$ for the "best" MSE matrix

$$\min_{\boldsymbol{w} \in \{0,1\}^N} f(\mathrm{E}\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\}), \qquad \hat{\boldsymbol{\theta}} = g(\boldsymbol{y})$$

- Here $f(\cdot)$ is some scalar performance measure and $g(\cdot)$ an estimator of $\boldsymbol{\theta}$

Application to Estimation

- Since the exact MSE is hard to express the CRB (inverse FIM) is used
- The CRB (inverse FIM) is a lower bound on any unbiased estimator

$$\mathrm{E}\{(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})^T\} \succeq \boxed{\mathbf{C}} = \boxed{\mathbf{F}}^{-1}$$

Cramer Rao bound (CRB) Fisher information matrix (FIM)

- The CRB/FIM has a number of advantages over using the MSE directly
  - Well-suited for offline design
  - Reveals local identifiability
  - Is invariant to the adopted algorithm
  - Exact error covariance matrix in the linear additive Gaussian case

Application to Estimation

- The CRB/FIM can be written in closed form as a function of $p(\boldsymbol{y};\boldsymbol{w},\boldsymbol{\theta})$

$$[\mathbf{F}(\boldsymbol{w},\boldsymbol{\theta})]_{k,l} = \mathrm{E}\left\{\frac{\partial p(\boldsymbol{y};\boldsymbol{w},\boldsymbol{\theta})}{\partial \theta_k}\frac{\partial p(\boldsymbol{y};\boldsymbol{w},\boldsymbol{\theta})}{\partial \theta_l}\right\}$$

- For a multivariate complex Gaussian $\mathcal{N}(\boldsymbol{\mu}(\boldsymbol{w},\boldsymbol{\theta}),\boldsymbol{C}(\boldsymbol{w},\boldsymbol{\theta}))$ we obtain

$$[\mathbf{F}(\boldsymbol{w},\boldsymbol{\theta})]_{k,l} = 2\mathrm{Re}\left\{\frac{\partial \boldsymbol{\mu}^H(\boldsymbol{w},\boldsymbol{\theta})}{\partial \theta_k}\boldsymbol{C}^{-1}(\boldsymbol{w},\boldsymbol{\theta})\frac{\partial \boldsymbol{\mu}^H(\boldsymbol{w},\boldsymbol{\theta})}{\partial \theta_k}\right\} + \mathrm{Tr}\left\{\boldsymbol{C}^{-1}(\boldsymbol{w},\boldsymbol{\theta})\frac{\partial \boldsymbol{C}(\boldsymbol{w},\boldsymbol{\theta})}{\partial \theta_k}\boldsymbol{C}^{-1}(\boldsymbol{w},\boldsymbol{\theta})\frac{\partial \boldsymbol{C}(\boldsymbol{w},\boldsymbol{\theta})}{\partial \theta_l}\right\}$$

- In case the observations are conditionally independent, the FIM is additive

$$\boldsymbol{F}(\boldsymbol{w},\boldsymbol{\theta}) = \sum_{n=1}^{N} w_n \boldsymbol{F}_n(\boldsymbol{\theta})$$

- This additive property even holds for dependent Gaussian observations!
- For nonlinear models and/or specific distributions, the FIM depends on $\boldsymbol{\theta}$
  - Optimize over a grid of $\boldsymbol{\theta}$ values
  - Use the Bayesian CRB, i.e., average the CRB over the prior pdf of $\boldsymbol{\theta}$

Application to Estimation

- We need a scalar measure based on the FIM
  - E-optimality (worst case error):

$$f(\boldsymbol{w}) := \lambda_{\max}\{\boldsymbol{F}^{-1}(\boldsymbol{w},\boldsymbol{\theta})\}$$

  - A-optimality (average error):

$$f(\boldsymbol{w}) := \operatorname{tr}\{\boldsymbol{F}^{-1}(\boldsymbol{w},\boldsymbol{\theta})\}$$

convex (SDP)

  - D-optimality (error volume):

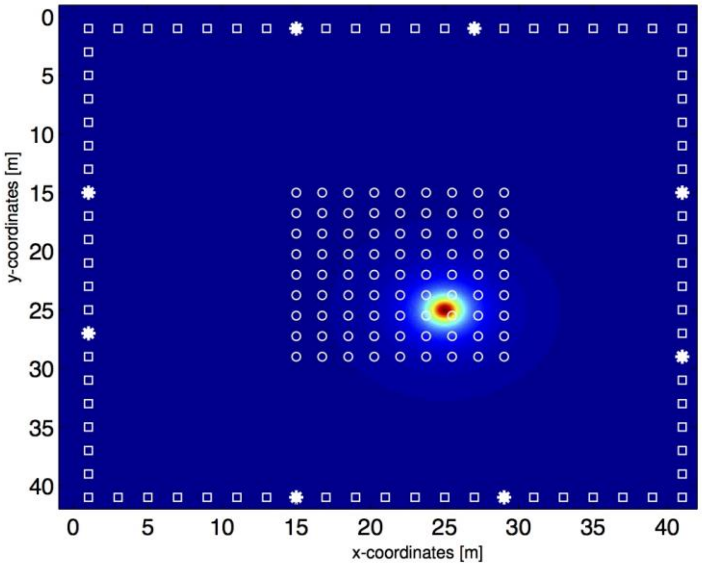
$$f(\boldsymbol{w}) := \ln\det\{\boldsymbol{F}^{-1}(\boldsymbol{w},\boldsymbol{\theta})\}$$

submodular

Target Localization

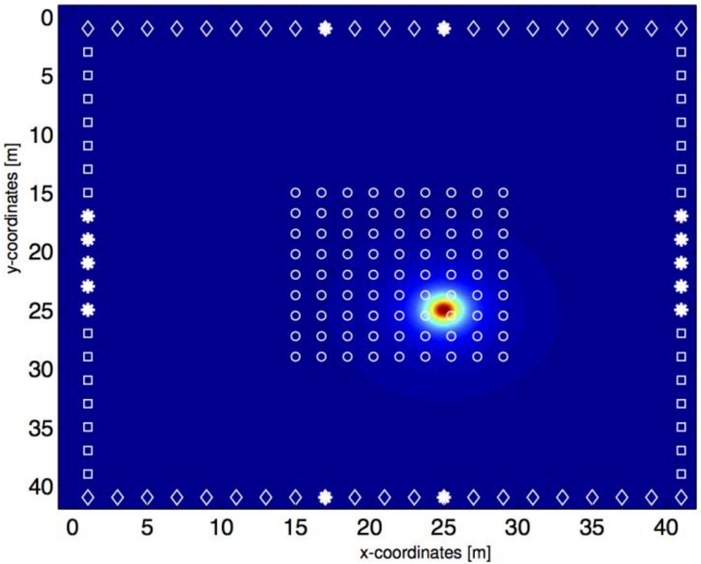
Application: target localization based on received signal strength (RSS)

Independent observations



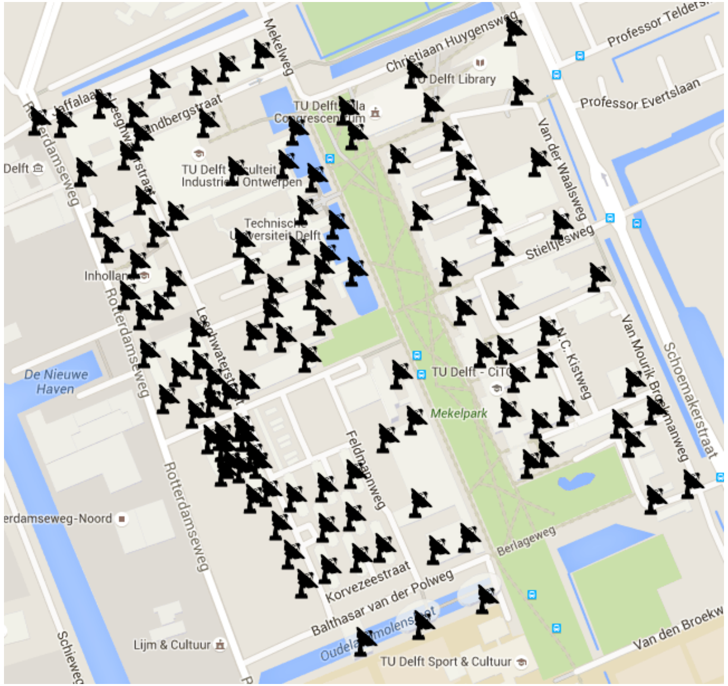
Dependent Gaussian observations

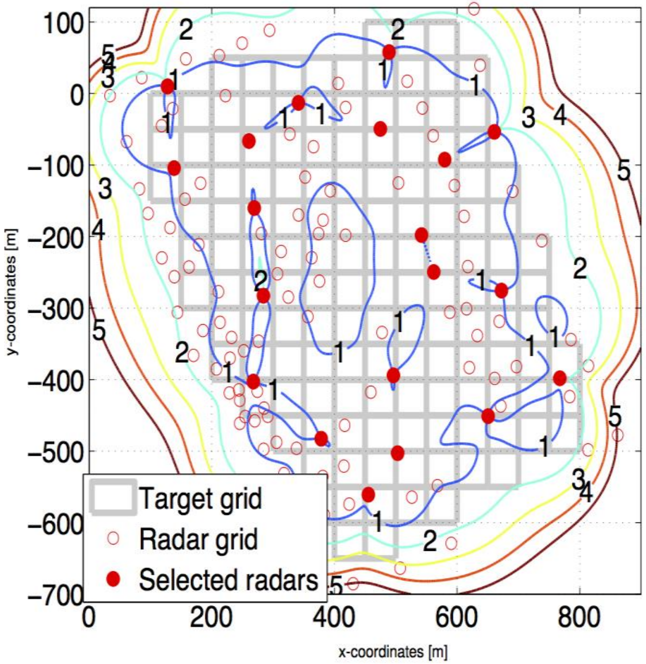
(horizontal sensors correlated)



Target Localization

Application: localization using multi-static FMCW radar





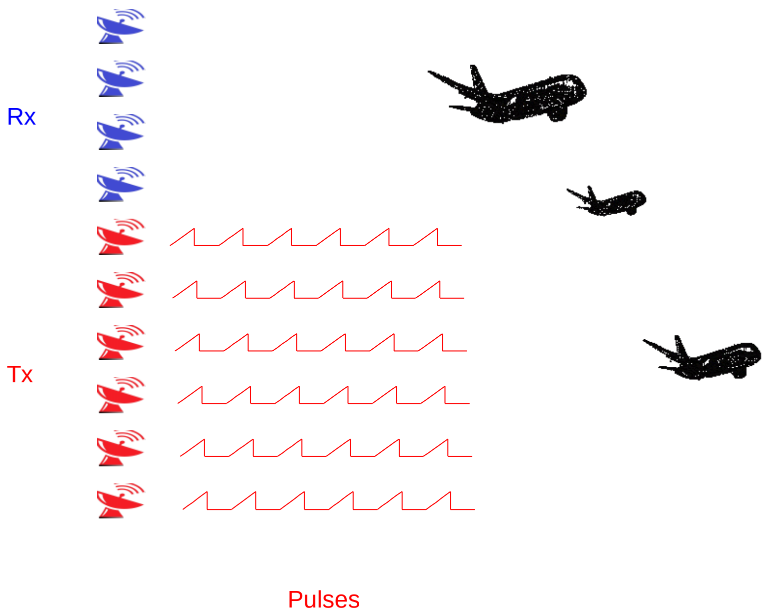
[Ivashko-Leus-Yarovoy, 2017]

Application to Array Design [Tohidi et al, 2019]

- The CRB focuses on local identifiability, i.e., main lobe width is optimized
  - Good if we have a single target
  - Not good if we want to focus on multiple targets
- Solutions
  - Add constraints to the sidelobes
  - Use a multi-target CRB
- We focus on a two-target CRB where the two targets can be anywhere
  - This CRB is manageable
  - The only unknown is the difference in the direction cosines $\cos\theta_1 - \cos\theta_2$
  - We grid this unknown or average the two-target CRB over it
  - As before, different optimality criteria and algorithms can be used

Colocated MIMO Radar [Tohidi et al, 2019]

- Goals are to optimize the number of antennas as well as pulses
- Unknowns are angle of arrival and radial velocity

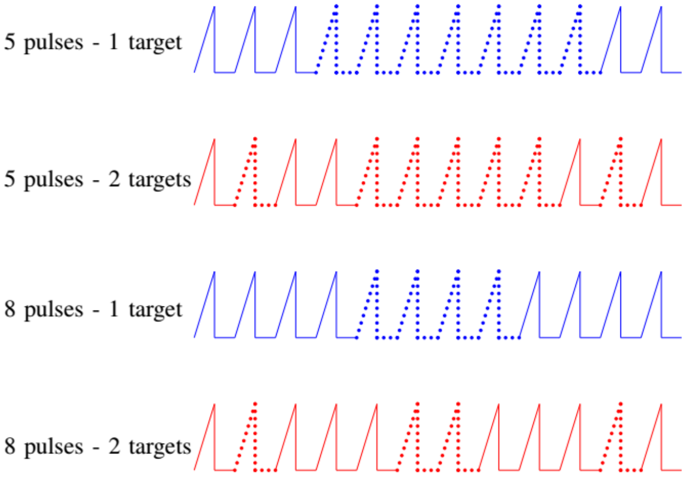


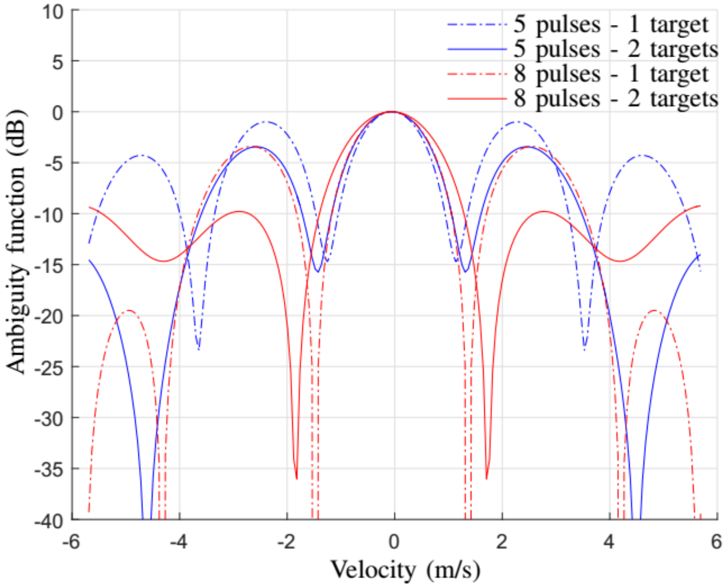
Colocated MIMO Radar

- We optimize the two-target CRB
  - Unknowns are differences in direction cosines and radial velocities
  - We consider a grid for both of them
- We consider E-optimality but other criteria can be used as well
- We consider the following cases
  - Single antenna – multiple pulses (velocity estimation)
  - Multiple antennas – single pulse (DoA estimation)
  - Combination of both
  - We use E-optimality
  - We use exhaustive search or convex optimization
  - Larger scenarios need to use greedy search

Colocated MIMO Radar

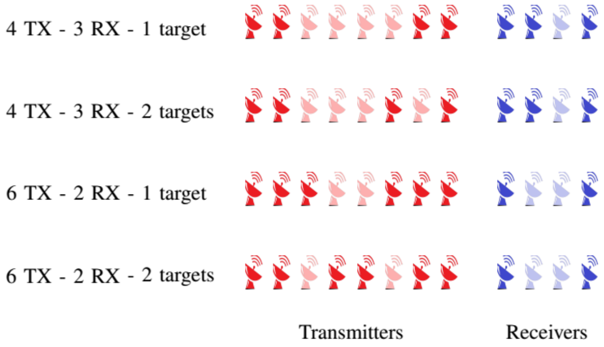
- Single antenna – multiple pulses

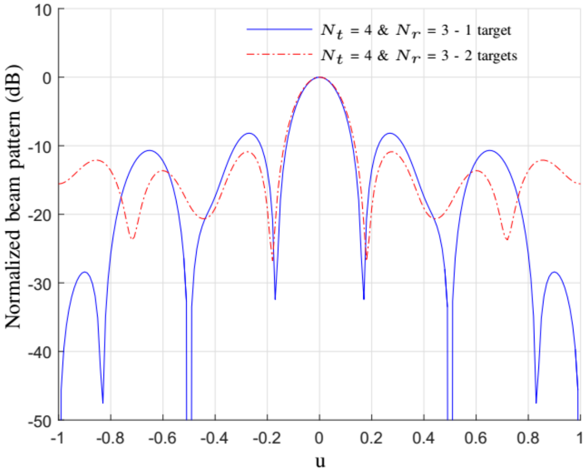


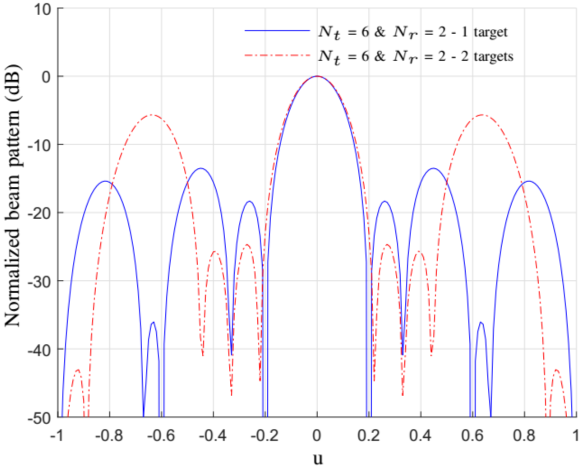


Colocated MIMO Radar

- Multiple antennas – single pulse

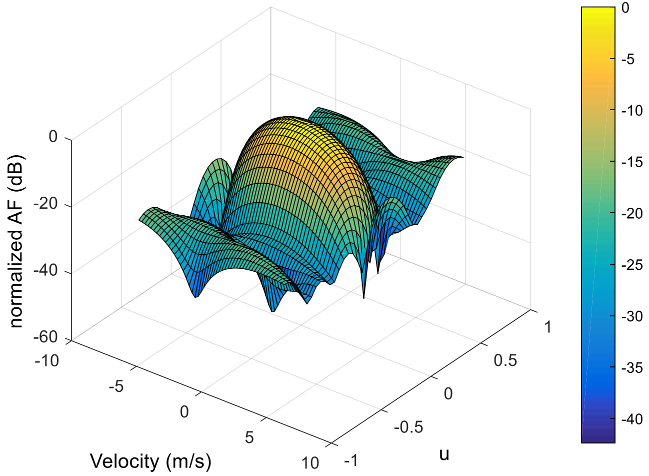


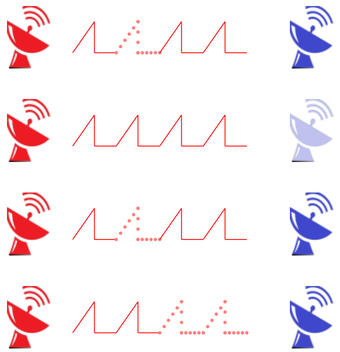




Colocated MIMO Radar

- Multiple antennas – multiple pulses





Extensions and Open Issues

- Can be extended for active radar [Aittomäki, 2017]
- Mutual coupling
- Calibration and robustness [Paulraj-Kailath, 1985] [Weiss-Friedlander, 1990]
- Machine learning for array design
- Data-driven parameter estimation
- Vector array processing [Nehorai-Paldi, 1994] [Ramamohan et al, 2017, 2018]
- Complexity and implementation aspects

Conclusions

- Compressive array can reduce “complexity” without much performance loss
- Standard DoA methods can be used with a price in the number of sources
- Covariance-based methods make up for this price
- Array design for compressive arrays
  - Generally based on compression rate and identifiability
  - Can be done for standard as well as covariance-based methods
- Performance-based sparse array design
  - Fits within the sparse sensing framework
  - Focus is mostly on one-target CRB or any other performance measure
  - Improvements can be obtained considering two-target CRB

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