Single-Pulse Estimation of Target Velocity on Planar Arrays

Costas A. Kokke[‡], Mario Coutiño[*], Richard Heusdens[†], Geert Leus[‡], Laura Anitori[*]
[*]Radar Technology, Netherlands Organisation for Applied Scientific Research, The Hague, The Netherlands
[†]Netherlands Defence Academy, Den Helder, The Netherlands
[‡]Circuits and Systems, Delft University of Technology, Delft, The Netherlands

***Abstract*—Doppler velocity estimation in pulse-Doppler radar is done by evaluating the target returns of bursts of pulses. While this provides convenience and accuracy, it requires multiple pulses. In adaptive and cognitive radar systems, the ability to adapt on consecutive pulses, instead of bursts, brings potential performance benefits. Hence, with radar transceiver arrays growing increasingly larger in their number of elements over the years, it may be time to re-evaluate how Doppler velocity can be estimated when using large planar arrays. In this work, we present variance bounds on the estimation of velocity using the Doppler shift as it appears in the array model. We also propose an efficient method of performing the velocity estimation and we verify its performance using Monte Carlo simulations.**

***Index Terms*—array signal processing, velocity estimation, Doppler processing, pulse-Doppler radar, Cramér-Rao bound**

I. INTRODUCTION

In pulse-Doppler radar systems, it is typical to estimate the radial velocity by evaluating multiple compressed pulses reflected by a target and estimating the linear phase change over time. It is known that the pulse repetition frequency (PRF), number of processed pulses, carrier wavelength and signal-to-noise ratio (SNR) after pulse compression are the primary contributors to the estimation performance. Cramér-Rao bounds (CRBs) of velocity estimation on arrays have been described in [1], [2], but the signal models used typically omit the Doppler velocity in the array response expression since this is considered negligible. When considering multiple pulses in a burst, the fast-time Doppler effect is also commonly omitted.

Numerous methods exist that aim to reach optimal velocity estimation on arrays [3]. However, such methods do not exploit the extra information given by using large receiver arrays, besides being able to scan more angles of arrival (AoA) and improved SNR. While these methods perform better than conventional pulse-Doppler processing, there may be more room for improvement considering a more complete signal model. Since modern radar systems are realized with increasingly larger arrays, both in terms of physical size and number of elements, such as the Sea-based X-Band Radar [4], it may be time to re-evaluate what this large amount of data allows us to do in terms of parameter estimation.

If the array response depends on the radial velocity, it might not be needed to evaluate a pulse train anymore. After all, the array response can be estimated using pulse compression of single pulses. Single pulse Doppler estimation is typically considered challenging or even infeasible since the phase changes due to Doppler shifts on the reflected waveform are so small they are negligible. However, the array response is an extra source of information on the radial velocity of the target that can be utilized to realize single pulse Doppler estimation.

Single pulse Doppler shift estimation can be useful for different reasons. First and foremost, it can be used in adaptive radar systems which rely on frequent and continuous updates of their surroundings to perform parameter optimization and resource allocation. The less 'out of date' the knowledge of the surrounding is, the better such a system should be able to perform [5]. Second, performing Doppler shift estimation on individual pulses from the array does not exclude one from performing the estimation on a burst of pulses as well. In this case, using the additional information granted by the array response, the performance of the Doppler shift estimation on the burst of pulses may improve. Third, [6] argues that single pulse Doppler shift estimation can be used as an alternative to motion compensation algorithms for fast-moving targets which move to different range cells within a full processing interval, or whose motion is not uniform. This particularly also applies to fast-moving targets at long ranges, where PRFs might be low and pulse lengths may be large.

In this contribution, we show the variance bounds of estimating the Doppler velocity on single pulses using only the array response, and using both the array response and the shifted radar waveform. We further show a comparison of bounds considering and not considering the array response as a source of information on the Doppler velocity, effectively comparing single pulse Doppler estimation to burst processing. We also introduce a method of estimating the Doppler velocity from the array response for the single target case, using a fast rank-1 matrix approximation and the fast Fourier transform (FFT). While the theoretical bound as well as the practical performance of the method show a large error, there may be a use for such approaches in scenarios of high SNR and/or using receive arrays with large physical aperture. Further, there may be applications in systems with lower wave travel speeds, such as sonar.

The work is part of a project funded by the Netherlands Organisation for Applied Scientific Research (TNO) and the Netherlands Defence Academy (NLDA).

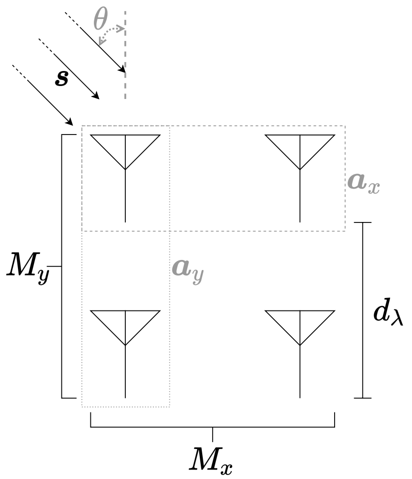


Fig. 1: Array configuration. In this example $M_x = M_y = 2$ and $M = M_x M_y = 4$.

II. Signal Model

Let us assume a target unimodulus signal $\boldsymbol{s}$ of length N samples, arriving on a uniform planar array (UPA) with array response $\boldsymbol{a}$, element spacing d_λ in wavelengths and M number of elements, see Fig. 1. The signal is critically sampled with sampling period T_s. The target is assumed to arrive from zero elevation and as such, we only describe its angle of arrival (AoA) with the azimuth angle θ. Let the Doppler shift due to the target speed be expressed by $\Delta\lambda = \left(1 + \frac{2v_r}{\mathrm{c}}\right)^{-1}$, which describes the multiplicative change in carrier wavelength λ_0, where v_r is the radial component of the target speed and c the speed of light. Additionally, let $f_d = \frac{2v_r}{\lambda_0}$ be the Doppler frequency, where λ_0 is the carrier wavelength.

We can then let $\boldsymbol{Y}$ be the received array data of a single reflected pulse, given by

$$\boldsymbol{Y} = \boldsymbol{a}\boldsymbol{s}^{\mathrm{T}} + \boldsymbol{V} \in \mathbb{C}^{M\times N}\,, \tag{1}$$

where, under the assumption of a 2-dimensional $M_x \times M_y$ array (thus $M = M_x M_y$), we have

$$\begin{aligned}
\boldsymbol{a} &= \boldsymbol{a}_x \otimes \boldsymbol{a}_y && \in \mathbb{C}^{M}\,, && (2)\\
\boldsymbol{a}_x &= \begin{bmatrix} z_x^0 & z_x^1 & \cdots & z_x^{M_x-1}\end{bmatrix}^{\mathrm{T}} && \in \mathbb{C}^{M_x}\,, &&\\
\boldsymbol{a}_y &= \begin{bmatrix} z_y^0 & z_y^1 & \cdots & z_y^{M_y-1}\end{bmatrix}^{\mathrm{T}} && \in \mathbb{C}^{M_y}\,, &&\\
z_x &= \exp\left\{\mathrm{j}\frac{2\pi d_\lambda}{\Delta\lambda}\sin\theta\right\}, && && (3)\\
z_y &= \exp\left\{\mathrm{j}\frac{2\pi d_\lambda}{\Delta\lambda}\cos\theta\right\}, && && (4)
\end{aligned}$$

and $\boldsymbol{s} \in \mathbb{C}^N$ is, as mentioned before, a vector describing the reflected unimodulus radar waveform as it arrives at the array, including the fast-time Doppler effect. Its elements are given by

$$s_n = \mathrm{e}^{\mathrm{j}(\phi_n + 2\pi f_d T_s n)}\,,$$

where n and ϕ_n are the sample index and the waveform phase at sample n, respectively. The radar cross section could be included explicitly, but the constant phase shift within a pulse is of no concequence to the discussion of this work and we normalize such that $|s_n| = 1$ for all n. The entries of $\boldsymbol{V} \in \mathbb{C}^{M\times N}$ are the noise realizations, which are drawn from a zero-mean Gaussian distribution with variance σ^2, and the noise is spatially and temporally uncorrelated. The pre-processing SNR is equal to σ^{-2}.

From (3) and (4), it should be clear that, if we can estimate the array response, the Doppler shift on the array response can be separated from the angle of arrival.

III. Cramér-Rao Bound for Velocity Estimation

Before we discuss our method of estimating the target radial velocity from the array response, let us first find the CRB of such an estimator, assuming θ is known. We start with the log-likelihood function of the observations, which can be expressed as

$$\begin{aligned}
L &= -MN\log\left(\pi\sigma^2\right) - \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\|\boldsymbol{y}_n - \boldsymbol{a}s_n\|_2^2\\
&= -MN\log\left(\pi\sigma^2\right) - \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\|\boldsymbol{v}_n\|_2^2\,,
\end{aligned}$$

where $\boldsymbol{y}_n$ and $\boldsymbol{v}_n$ are the columns of $\boldsymbol{Y}$ and $\boldsymbol{V}$, respectively.

Given the unknown parameter vector $\boldsymbol{\xi} = \begin{bmatrix}\sigma^2 & v_r\end{bmatrix}$, the CRB for velocity estimation is given by the bottom-right entry of the inverse of the Fisher information matrix (FIM) given by

$$\boldsymbol{F}_{\boldsymbol{\xi}} = -\,\mathrm{E}\left\{\begin{bmatrix}\frac{\partial^2 L}{\partial(\sigma^2)^2} & \frac{\partial^2 L}{\partial\sigma^2\partial v_r}\\ \frac{\partial^2 L}{\partial v_r\partial\sigma^2} & \frac{\partial^2 L}{\partial v_r^2}\end{bmatrix}\right\}. \tag{5}$$

Since $\mathrm{E}\left\{\frac{\partial^2 L}{\partial\sigma^2\partial v_r}\right\} = \mathrm{E}\left\{\frac{\partial^2 L}{\partial v_r\partial\sigma^2}\right\} = 0$, we need only concern ourselves with deriving $F_{v_r} = -\,\mathrm{E}\left\{\frac{\partial^2 L}{\partial v_r^2}\right\}$.

We can then derive

$$\frac{\partial L}{\partial v_r} = \frac{2}{\sigma^2}\sum_{n=0}^{N-1}\mathrm{Re}\left\{s_n^*\dot{\boldsymbol{a}}^{\mathrm{H}}\boldsymbol{v}_n + \dot{s}_n^*\boldsymbol{a}^{\mathrm{H}}\boldsymbol{v}_n\right\}\,,$$

where $\dot{\boldsymbol{a}} = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}v_r}$ and $\dot{s}_n = \frac{\mathrm{d}s_n}{\mathrm{d}v_r}$. Next,

$$\begin{aligned}
-\,\mathrm{E}\left\{\frac{\partial^2 L}{\partial v_r^2}\right\} = \frac{2}{\sigma^2}\Bigg(&N\dot{\boldsymbol{a}}^{\mathrm{H}}\dot{\boldsymbol{a}} + \sum_{n=0}^{N-1}M\|\dot{s}_n\|_2^2\\
&+ 2\mathrm{Re}\left\{s_n^*\dot{\boldsymbol{a}}^{\mathrm{H}}\boldsymbol{a}\dot{s}_n\right\}\Bigg)\,,
\end{aligned}$$

gives the relevant diagonal entry of $\boldsymbol{F}_{\boldsymbol{\xi}}$. We can work this out a little further leading to the expressions

$$\begin{aligned}
\dot{\boldsymbol{a}}^{\mathrm{H}}\dot{\boldsymbol{a}} = \frac{8\pi^2 d_\lambda^2 M}{\mathrm{c}^2}\Bigg(&\frac{1}{3}(M_x-1)(2M_x-1)\sin^2\theta\\
&+ \frac{1}{3}(M_y-1)(2M_y-1)\cos^2\theta\\
&+ \frac{1}{2}(M_x-1)(M_y-1)\sin 2\theta\Bigg)\,,
\end{aligned}$$

$$\sum_{n=0}^{N-1} M\|\dot{s}_n\|_2^2 = \frac{8\pi^2 M T_s^2}{3\lambda_0^2} N(N-1)(2N-1)\,,$$

$$\sum_{n=0}^{N-1} 2\mathrm{Re}\big(s_n^* \dot{\boldsymbol{a}}^{\mathrm{H}} \boldsymbol{a} \dot{s}_n\big) = \frac{8\pi^2 d_\lambda T_s M}{\mathrm{c}\lambda_0} N(N-1) \times \big(M_x(M_x-1)\sin\theta + M_y(M_y-1)\cos\theta\big)\,.$$

From these equations we find the CRB to be

$$\mathrm{CRB}(\hat{v}_r) = \frac{\sigma^2}{16\pi^2 MN(\alpha+\beta+\gamma)}\,, \tag{6}$$

where

$$\alpha = \frac{d_\lambda^2}{\mathrm{c}^2}\bigg(\frac{1}{3}(M_x-1)(2M_x-1)\sin^2\theta + \frac{1}{3}(M_y-1)(2M_y-1)\cos^2\theta + \frac{1}{2}(M_x-1)(M_y-1)\sin 2\theta\bigg)\,, \tag{7}$$

$$\beta = \frac{T_s^2}{3\lambda_0^2}(N-1)(2N-1)\,, \tag{8}$$

$$\gamma = \frac{d_\lambda T_s}{\mathrm{c}\lambda_0}(N-1) \times \big(M_x(M_x-1)\cos\theta - M_y(M_y-1)\sin\theta\big)\,.$$

These three summands each correspond to unique contributions:

- α describes the contribution of the velocity term in the array response.
- β describes the contribution of the fast-time Doppler in the reflected waveform.
- γ is a cross-term due to the two sources of Doppler in our model.

By comparing (7) and (8), it becomes clear that increasing the listening time, NT_s, is far more beneficial in lowering the CRB than increasing the number of antennas. This is further illustrated by Fig. 2a where

$$\mathrm{Gain} = \frac{\mathrm{CRB}(\hat{v}_r)|_{\alpha=0,\gamma=0}}{\mathrm{CRB}(\hat{v}_r)} \tag{9}$$

is plotted, indicating the improvement in CRB when the Doppler effect on the array is considered instead of neglected. This confirms that neglecting the Doppler effect on the array response when multiple pulses are available is valid since the listening time in that scenario is dictating the estimation performance bound. As a result, it looks like there are few scenarios where considering the Doppler effect on the array response in the signal model is worth considering. One example of when it may be beneficial still to consider the Doppler effect in the array response model is when a waveform with range-Doppler coupling is used, such as linear frequency modulated waveforms.

Also, increasing the element spacing d_λ beyond $\frac{1}{2}$ increases the contribution of the array information for velocity estimation. While this should result in spatial aliasing, it can be mitigated by having sub-arrays with smaller element spacing. This way of distributing multiple antennas over a large aperture by considering multiple subarrays is similar to how multiple pulses can be distributed over a larger period of time. The bound for different amounts of antennas with different element spacings is shown in Fig. 2b.

IV. Fast Single Pulse Velocity Estimation

From Section III, it seems of little use to attempt to estimate a target velocity from the array response in general. However, it will not always be possible to perform processing on the target reflected pulse on all array elements, due to hardware constraints. If one would still desire to perform single pulse velocity estimation, there may be no other choice but to perform the estimation using the array response estimate.

For the single target, single pulse case, we can perform a rank-1 estimation of the array response matrix $\boldsymbol{A} = \boldsymbol{a}_x \boldsymbol{a}_y^{\mathrm{T}}$. We obtain $\hat{\boldsymbol{A}}$ by performing a matched filter operation on each array element individually. Let $\hat{\boldsymbol{a}}$ be the estimated array response given by $\hat{\boldsymbol{a}} = \boldsymbol{Y}\boldsymbol{h}_{\mathrm{MF}}$, where

$$\boldsymbol{h}_{\mathrm{MF}} = \frac{\boldsymbol{s}^*}{\|\boldsymbol{s}\|_2^2} = \frac{\boldsymbol{s}^*}{N}\,. \tag{10}$$

We can then restructure the obtained estimate $\hat{\boldsymbol{a}}$ to obtain

$$\hat{\boldsymbol{A}} = \hat{\boldsymbol{a}}_x \hat{\boldsymbol{a}}_y^{\mathrm{T}}\,, \tag{11}$$

i.e., the resulting matrix resembles the topography of the array.

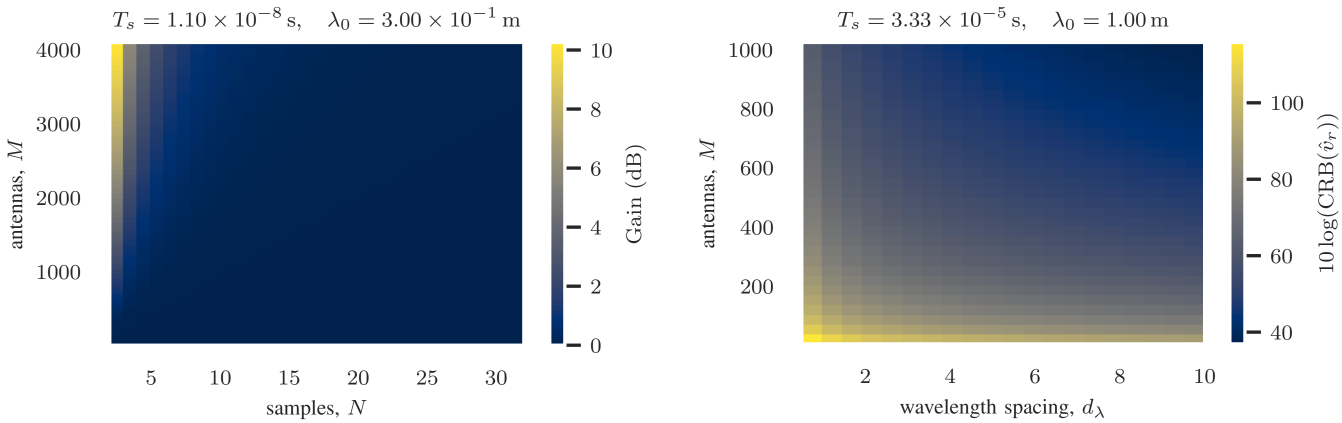
Once we have estimated this array response matrix, we perform a compact singular value decomposition (cSVD)[1] on it to obtain the left and right singular vectors associated with the largest singular value: $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$, respectively. By (11) we can then say that $\boldsymbol{u}_1 \sim \boldsymbol{a}_x$ and $\boldsymbol{v}_1 \sim \boldsymbol{a}_y$ up to some scalar ambiguity. We then need to find the dominant normalized frequency of these vectors, which should both approximate $d_\lambda\big(1+\frac{2v_r}{\mathrm{c}}\big)$.

To find these dominant frequencies, we propose to use an FFT. It should be noted that to obtain a good estimation resolution, one would either need to use a very long FFT, apply methods such as [7] or perform a short local search after finding a coarse estimate. We will not further describe such, already well-documented, extensions here and will simply use long FFT lengths to obtain a sufficiently finely sampled spectrum.

When the planar array is not uniform, but if the planar array response vector can be written as the Kronecker product of two non-uniform linear array response vectors (similar to (2)), then a type-II nonuniform discrete Fourier transform [8] can be employed instead of an ordinary FFT.

The frequency estimates can be combined to find the AoA and velocity. To keep the discussion concise, we assume the AoA known and the frequency estimate are combined in a weighted average according to the array dimensions. The velocity estimation procedure is summarized by Alg. 1.

[1] A cSVD only calculates an amount of singular values, and left and right singular vectors up to the given rank of the input matrix.



(a) Comparison of the CRBs when neglecting and including the velocity in the array response expression. The gain, given by (9), indicates how much lower the CRB of including the velocity in the array response is.

(b) The bound in (6) for different amounts of antennas with different element spacings. In this figure, $\frac{N}{\sigma^2} = 30\,\mathrm{dB}$.

Fig. 2: Cramér-Rao bound analysis.

Require: $\boldsymbol{Y} \in \mathbb{C}^{M \times N}, \boldsymbol{h}_{\mathrm{MF}} \in \mathbb{C}^N, M_x, M_y, \theta$

$\hat{\boldsymbol{a}} \leftarrow \boldsymbol{Y}\boldsymbol{h}_{\mathrm{MF}}$ ▷ using (10)

$\hat{\boldsymbol{A}} \leftarrow \mathrm{reshape}(\hat{\boldsymbol{a}})$ ▷ according to (11)

$\boldsymbol{u}_1, \sigma_1, \boldsymbol{v}_1 \leftarrow \mathrm{cSVD}\left(\hat{\boldsymbol{A}}\right)$

$\psi_x \leftarrow \arg\max\ \mathrm{FFT}(\boldsymbol{u}_1), \quad \psi_y \leftarrow \arg\max\ \mathrm{FFT}(\boldsymbol{v}_1)$

$\hat{v}_r \leftarrow \frac{c}{2}\left(\frac{M_x \psi_x}{(M_x+M_y)d_\lambda \sin\theta} + \frac{M_y \psi_y}{(M_x+M_y)d_\lambda \cos\theta} - 1\right)$

Alg. 1: Fast single pulse velocity estimation procedure utilizing the cSVD and FFT.

V. Numerical Results

To verify the proposed method and quantify its performance we have performed a number of Monte-Carlo simulations. For the sake of brevity, the following parameters are fixed for all simulations:

- Number of trials, $N_{\text{trials}} = 100$
- Number of pulses, $K = 1$
- Angle of arrival, $\theta = \frac{\pi}{4}$
- Radial velocity, $v_r = 0$
- Array element spacing in wavelengths, $d_\lambda = \frac{1}{2}$
- Matched filter, $\boldsymbol{h}_{\mathrm{MF}} = \boldsymbol{s}^*|_{f_d=0}$
- FFT length, $N_{\text{fft}} = 2^{24}$

For convenience, we set $\left(\frac{S}{N}\right)_{\mathrm{MF}} = \frac{N}{\sigma^2}$ as our sweeping parameter in the simulations. This quantity can be considered the SNR after matched filter. In practice, this SNR may drop when there is a mismatch between target velocity and the specific matched filter(s) that are used. As discussed in Section IV, the length of the FFT here is large to obtain a sufficiently finely sampled spectrum. This number can be reduced in practice while maintaining resolution, as discussed earlier.

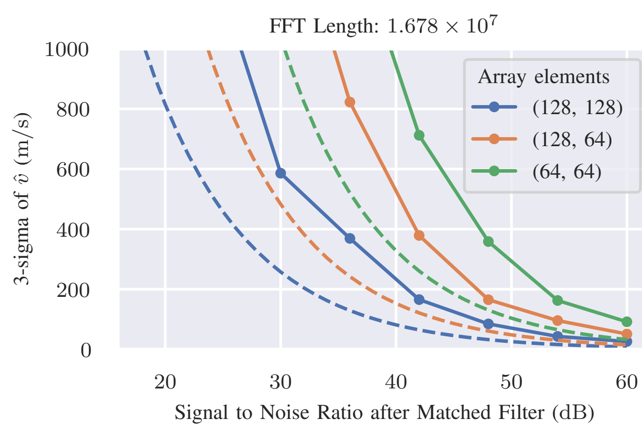


Fig. 3: The variance the velocity estimation and the CRB. Dotted curves indicate the CRBs, and the •s indicate the estimator performance of Alg. 1.

Fig. 3 summarizes the results of the Monte-Carlo simulations by showing the 3-sigma values of the simulation outputs for a few different arrays and $\left(\frac{S}{N}\right)_{\mathrm{MF}}$ values. There is a considerable difference in the performance of the estimator and the CRB (a factor > 2), likely due to the multi-stage approach and the considerable sensitivity of the velocity parameter.

VI. Conclusions

We have presented a method of estimating the Doppler velocity on uniform planar arrays using the outputs of matched filters, i.e., the estimated array response. This method reduces the size of the problem and uses common algorithms that are efficiently implementable. However, the method does not achieve the presented variance bound. By inspection of the variance bound in Figs. 2 and 3, we conclude that such methods will be primarily useful in (sparse) arrays with large apertures and/or for long Doppler-tolerant pulses, with applications such as early warning systems for very fast targets, where an order-of-magnitude estimation is sufficient.

We can show through the variance bounds that the gain from using the array response as an extra source of information on the velocity is negligible in most other scenarios. In scenarios where this extra source of information might be helpful, a method to estimate the velocity from the full data model, instead of only the array response or only the slow-time pulses, may be beneficial. This remains as future work.

Our signal model results in a similar data structure as some models used in MIMO radar estimation methods [9], [10]. They present methods to solve the rootMUSIC algorithm for two variables. Such estimation methods may be adapted to the problem presented in this work, though their complexity, especially for significantly large arrays, may be too high. Conversely, our method may be applied to find the angle of departure and AoA in the single target case of [9], as a low complexity alternative to double rootMUSIC.

Similarly, the rootMUSIC polynomial could be solved through λ-matrix latent-root-finding [11], [12]. While such methods were considered, the specific form of our problem did not allow for low-complexity methods of solving such polynomials since our λ-matrix is neither regular nor coregular.

References

[1] A. Dogandzic and A. Nehorai, "Cramer-Rao bounds for estimating range, velocity, and direction with an active array," *IEEE Transactions on Signal Processing*, vol. 49, no. 6, pp. 1122–1137, Jun 2001.

[2] J. Ward, "Cramer-Rao bounds for target angle and Doppler estimation with space-time adaptive processing radar," in *Conference Record of The Twenty-Ninth Asilomar Conference on Signals, Systems and Computers*, vol. 2, Oct 1995, pp. 1198–1202 vol.2.

[3] H. Abeida, Q. Zhang, J. Li, and N. Merabtine, "Iterative sparse asymptotic minimum variance based approaches for array processing," *IEEE Transactions on Signal Processing*, vol. 61, no. 4, pp. 933–944, Feb 2013.

[4] "The sea-based X-band radar-1 (SBX-1)." [Online]. Available: https://www.naval-technology.com/projects/sea-based-x-band-radar-1-sbx-1/

[5] J. R. Guerci, *Cognitive radar: the knowledge-aided fully adaptive approach*, 2nd ed., ser. Artech House radar series. Artech House, 2020.

[6] S. D. Blunt, A. K. Shackelford, and K. Gerlach, "Single pulse imaging," in *2006 International Waveform Diversity Design Conference*, Jan 2006, pp. 1–5.

[7] E. Hoyer and R. Stork, "The zoom FFT using complex modulation," in *ICASSP '77. IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 2, 1977, pp. 78–81.

[8] J. Keiner, S. Kunis, and D. Potts, "Using nfft 3—a software library for various nonequispaced fast fourier transforms," *ACM Trans. Math. Softw.*, vol. 36, Aug 2009.

[9] M. L. Bencheikh, Y. Wang, and H. He, "Polynomial root finding technique for joint DOA DOD estimation in bistatic MIMO radar," *Signal Processing*, vol. 90, no. 9, pp. 2723–2730, Sep 2010.

[10] J. Lee, J. Park, and J. Chun, "Weighted two-dimensional root MUSIC for joint angle-Doppler estimation with MIMO radar," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 55, no. 3, pp. 1474–1482, Jun 2019.

[11] J. E. Dennis, J. F. Traub, and R. P. Weber, "On the matrix polynomial, lambda-matrix and block eigenvalue problems," Carnegie-Mellon University, Computer Science Department, Tech. Rep., 1971.

[12] B. Bekhiti, *Algorithmic Linear Algebra and Computational Methods in Regular Matrix Polynomials*. Saad Dahlab University, Dec 2020.