TNO

CENTRAAL LABORATORIUM

COMMUNICATION NO. **259**

HOLE CALCULATIONS FOR AN OBOE

by

C. J. NEDERVEEN

and

A. DE BRUIJN *)

*) Technological University, Delft

Reprinted from

Acustica **18** (1) (1967) 47 - 57

CENTRAAL LABORATORIUM TNO, P.O. BOX 217, DELFT, THE NETHERLANDS

ACUSTICA
S. HIRZEL VERLAG · STUTTGART
Vol. 18 1967 Heft 1

Hole Calculations for an Oboe

by C. J. Nederveen
Central Laboratory TNO, Delft
and A. de Bruijn
Technological University of Delft

Summary

Methods are derived to calculate the required positions and sizes of holes in a conical woodwind. With a simple algebraic expression, the holes are located, independently of one another. Only the frequencies for the opened and the closed positions need to be specified.

An approximative solution of the wave equation in a conical tube is derived including viscous and thermal damping along the walls and a complex impedance at the entrance of the tube. Pitch variations due to these effects are determined. Corrections are introduced for departures from the ideal cone and for the row of closed side holes.

The difference between calculated and measured hole positions is plotted, and the separate points scatter 0.1 to 0.2 semitone around a smooth curve drawn through them. The deviation from zero of the smooth curve indicates that the low register may be expected to be flat with respect to the high register.

Berechnung der Löcher an einer Oboe

Zusammenfassung

Es werden Verfahren zur Berechnung der erforderlichen Lage und Größe der Löcher in einem konischen Holzblasinstrument hergeleitet. Mit Hilfe eines einfachen algebraischen Ausdruckes wird die Lage der Löcher unabhängig voneinander bestimmt. Nur die Frequenzen für die geöffneten und geschlossenen Löcher müssen angegeben werden.

Unter Berücksichtigung von viskoser und thermischer Dämpfung an den Wänden und einer komplexen Impedanz am Eingang wird eine Näherungslösung der Wellengleichung für ein konisches Rohr hergeleitet. Die durch solche Einflüsse hervorgerufene Änderung der Tonhöhe wird bestimmt. Schließlich werden Korrekturen für Abweichungen vom idealen Konus und für die Reihe der geschlossenen Seitenlöcher eingeführt.

Der Unterschied in der errechneten und gemessenen Lage der Löcher wird aufgetragen, wobei die einzelnen Punkte um 0,1 bis 0,2 Halbtöne um eine durch sie gezeichnete glatte Kurve streuen. Die Abweichung dieser Kurve von Null läßt darauf schließen, daß das niedrige Register in bezug auf das hohe flach sein wird.

Calculs des trous pour un hautbois

Sommaire

On a établi des méthodes pour calculer les positions requises et les dimensions du trou dans les bois coniques. Par une simple expression algébrique, on détermine la position des trous, indépendamment les uns des autres. Seules les fréquences pour les positions ouvertes et fermées doivent être précisées.

On tire une solution approximative de l'équation des ondes dans un tube conique, comprenant l'amortissement visqueux et thermique le long des parois et une impédance complexe à l'entrée du tube. On détermine les variations de tons dues à ces influences. On présente des corrections pour les écarts du cône idéal et pour la rangée des trous en position fermée.

On fait le relevé de la différence entre les positions calculées et mesurées des trous et on trace une courbe lissée au milieu de la dispersion des points expérimentaux de 0,1 à 0,2 demi-tons. L'écart au zéro de la courbe lissée indique qu'on peut s'attendre à ce que le registre grave soit plat par rapport au registre aigu.

1. Introduction

Though the behaviour of woodwind instruments is far from unterstood, a gradual increase in knowledge may be observed. A few studies have appeared [1] to [8], but this subject seems to lag behind developments in other fields of acoustics [9], [10]. The present paper is a continuation of the one [7] on hole calculations for a clarinet, and concerns analogous investigations for an oboe. Whereas the clarinet is a cylindrical instrument, the oboe is essentially conical. It will appear that the method for the clarinet can be adapted to conical instruments. This means that each hole can be calculated separately, and independently of all others, only when the frequencies for both the open and the closed position of the hole are specified.

A number of corrections to these calculations need to be introduced which, all together, lower the pitch about one half to a whole tone. This makes theory and practice coincide within acceptable limits. When these corrections are not applied, the oboe is 2 cm to 5 cm shorter than expected from simple theory, a phenomenon already observed by REDFIELD [11].

Roughly speaking, the internal diameter of an oboe follows the course of a truncated cone closed at its narrow end by two flat wooden platelets, the reeds. By pressing air between these reeds, the player excites the air column into longitudinal vibrations [3]. These vibrations are damped by friction along the walls and by radiation from the holes and the open end.

RUSSELL [5] measured very accurately the diameters of a great number of oboes, and showed that the conicity of oboes is far from perfect. An example is presented in Fig. 1, where a plot is given of the diameter as a function of the distance to the reed tip for a Mönnich oboe. Near top and bottom, considerable departures from a cone are observed. Also the middle part shows deviations, though on a much smaller scale. To vizualize these, a plot is made of the deviations from a "standard cone", as introduced by RUSSELL [5]. The diameter of this standard cone is 0.025 times the distance to the top of the cone. In Fig. 2, the quantity $d-0.025(l_t-l_A)$ is plotted versus (l_t-l_A); the diameter is denoted by d, l_t is the distance to the top of the instrument without staple and reed, and l_A is an auxiliary constant to keep the curves separated. Three curves are given, (1) for a Marigaux (no. 3614, Paris, bought new in 1965), l_A = zero; (2) for a Lorée (no. AS 17/AAS 17, Paris, 1954), l_A = 25 mm; (3) for a Mönnich (no. 1055, Markneukirchen, 1962), $l_A = 50$ mm.

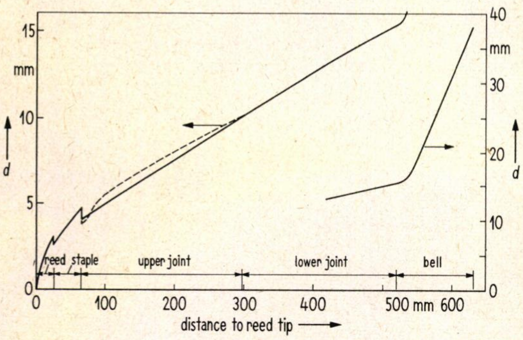

Fig. 1. Diameter d as a function of distance to the reed tip for a Mönnich oboe. Dashed line is the ideal bore according to RUSSELL [5]. The rapidly flaring bell is plotted on another scale.

The points refer to measurements obtained by carefully inserting cylindrical calibers of perspex in the wide end of the tube. Approximate curves or straight lines are drawn through the points. Oblique lines indicating the diameters are drawn, so that the diameter can be read at every place.

From studying Fig. 2, we conclude that besides some scatter (which is not due to measurement errors) each of the oboes shows a different trend in its diameter. A constriction at the top is present in all three of them. Somewhere near the joint of upper and lower part, the conicity changes rather abruptly. The Mönnich shows a wavery irregularity in the middle.

Before dealing with all these perturbations, we shall now determine the resonance frequency of a perfect cone where reed action, wall friction and radiation are taken into account.

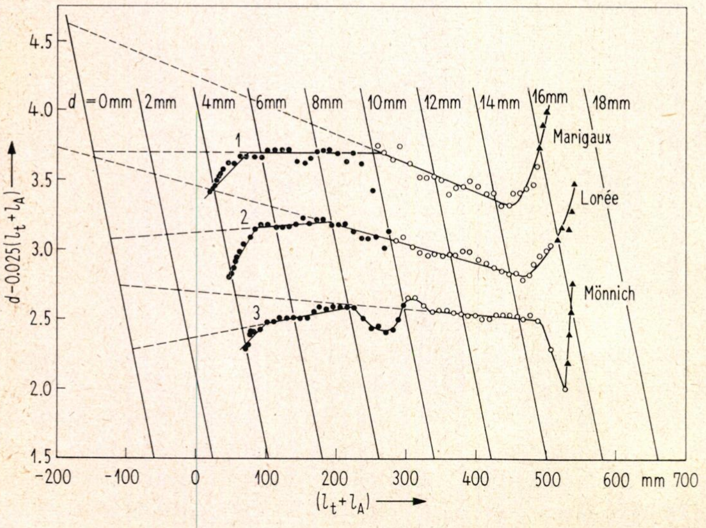

Fig. 2.
For three current oboes, the difference with RUSSELL's Standard Cone $d=l/40$ is plotted against (l_t-l_A). l_t is the distance to the top of the instrument, l_A is a constant for keeping the curves separated. For the Marigaux, $l_A=0$, for the Lorée $l_A=25$ mm, for the Mönnich $l_A=50$ mm.
● = upper part,
○ = lower part,
▲ = bell.

2. Theory of damped waves in a conical tube

At the walls, two effects can be distinguished: First, the gas layer at the wall does not move, so the viscosity of the gas will cause viscous friction. Secondly, the heat conductivity of the wall is good as compared with that of the gas, so it has a constant temperature, whereas compressions and expansions in the gas are adiabatic, and are accompanied by temperature fluctuations. An exact theory taking into account both effects simultanuously was given by Kirchhoff. For musical instruments, the thermal and viscous influence of the walls is only found in a thin layer along the walls. In this special case, the results can be described by correcting the density with a viscous term and correcting the bulk modulus with a term for thermal forces. It is now seen that after all a rather simple solution is possible; it will be given here.

As only studies are known for a cylindrical tube [12], [13], [14], we shall start with this tube and later modify the formulas to fit the conical tube. The equations of motion and continuity for longitudinal air motions in a cylindrical tube are given by Zwikker and Kosten [14] and read in a slightly modified notation:

$$\operatorname{grad} p = -\mathrm{j}\,\omega\,\varrho\,u \left[1 - \frac{2}{H a}\,\frac{J_1(H a)}{J_0(H a)}\right]^{-1} \tag{1}$$

$$\operatorname{div} u = -\frac{\mathrm{j}\,\omega}{\gamma\,p_0}\left[1 + \frac{2(\gamma-1)}{H' a}\,\frac{J_1(H' a)}{J_0(H' a)}\right] p \tag{2}$$

where:

p, p_0	=	excess respectively equilibrium pressure,
ω	=	2π times frequency,
ϱ	=	equilibrium density,
u	=	particle velocity,
γ	=	ratio of specific heats,
J_0, J_1	=	Bessel functions of zero respectively first order,
H	=	$\sqrt{-\mathrm{j}\,\omega\,\varrho\,\eta}$,
H'	=	$\sqrt{-\mathrm{j}\,\omega\,\gamma\varrho\,C_p/\lambda}$,
η	=	coefficient of viscosity,
λ	=	thermal conductivity,
a	=	tube radius,
C_p	=	heat capacity at constant pressure.

A development in series gives [15]

$$J_1(\beta)/J_0(\beta) = \mathrm{j} + 1/(2\beta) + 11\,\mathrm{j}/(64\,\beta^2) + \dots \tag{3}$$

When a is larger than thermal and viscous penetration depths, a good approximation is $J_1/J_0 = \mathrm{j}$. Herewith, the equations are simplified considerably. Elimination of u between eqs. (1) and (2) gives the wave equation for longitudinal waves in the z-direction:

$$\partial^2 p/\partial z^2 + k^2[1 + (1-\mathrm{j})/Q]\,p = 0 \tag{4}$$

where: $k = \omega/c$ is the wave number, and c is the sound velocity in free space. The resonator quality Q is found from

$$Q^{-1} = \sqrt{2\,\eta\,\omega/\gamma\,p_0}\,[1 + (\gamma-1)\,(\lambda/C_p\,\eta)^{\frac{1}{2}}]/k\,a\,. \tag{5}$$

In a conical tube, the radius a is dependent on the distance r to the top according to $a = \Theta\,r$. The conicity Θ is small in musical instruments, and the wave fronts may be assumed to be flat. The tube radius is small as compared to the wavelength everywhere. Eqs. (1) and (2), given for a cylindrical tube maintain their validity for a conical pipe, provided the dependence of the radius on r is inserted. Just as before we assume $J_1/J_0 = \mathrm{j}$ and we get:

$$\operatorname{grad} p = -\mathrm{j}\,\omega\,\varrho(1 + D_\mathrm{v}/x)\,u \tag{6}$$

$$\operatorname{div} u = -(\mathrm{j}\,\omega/\gamma\,p_0)\,(1 + D_\mathrm{t}/x)\,p \tag{7}$$

where: $x = k\,r$ and

$$D_\mathrm{v} = (1-\mathrm{j})\,D_\mathrm{v}' = (1-\mathrm{j})\,(2\,\omega\,\eta/\gamma\,p_0)^{\frac{1}{2}}\,\Theta^{-1}$$
$$D_\mathrm{t} = (1-\mathrm{j})\,D_\mathrm{t}' = (1-\mathrm{j})\,(\gamma-1)\,(\lambda/C_p\,\eta)^{\frac{1}{2}}\,D_\mathrm{v}' \tag{8}$$

D_v' and D_t' are real quantities and will be used later.

In the following derivation, these quantities are assumed to be small everywhere, so that quadratic and higher powers can be neglected. For a conical tube $\operatorname{grad} p = \mathrm{d}p/\mathrm{d}r$ and $\operatorname{div} u = \mathrm{d}r^2\,u/r^2\,\mathrm{d}r$. Elimination of u between eqs. (6) and (7) then gives:

$$\frac{\partial^2 p}{\partial x^2} + \left(\frac{2}{x} + \frac{D_\mathrm{v}}{x^2}\right)\frac{\partial p}{\partial x} + \left(1 + \frac{D_\mathrm{vt}}{x}\right) p = 0 \tag{9}$$

where $$D_\mathrm{vt} = D_\mathrm{v} + D_\mathrm{t}\,. \tag{10}$$

Assume a solution

$$\hat{p} = (p/x)\,[1 + g(x)]\exp(\pm\mathrm{j}\,x)\,. \tag{11}$$

Substitution into eq. (9) and multiplication with $\exp(\pm\mathrm{j}\,x)$ gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}g}{\mathrm{d}x}\exp(\pm 2\,\mathrm{j}\,x)\right] = -\left(\frac{D_\mathrm{vt}}{x} \pm \mathrm{j}\,\frac{D_\mathrm{v}}{x^2} - \frac{D_\mathrm{v}}{x^3}\right)\exp(\pm 2\,\mathrm{j}\,x)\,.$$

Integrating two times, one gets, apart from constants:

$$\mathrm{d}g/\mathrm{d}x = -D_\mathrm{vt}\,\mathrm{Ei}(\pm 2\,\mathrm{j}\,x)\exp(\mp 2\,\mathrm{j}\,x) - D_\mathrm{v}/2\,x^2 \tag{12}$$

$$g = \mp\mathrm{j}\,D_\mathrm{vt}[\mathrm{Ei}(\pm 2\,\mathrm{j}\,x)\exp(\mp 2\,\mathrm{j}\,x) - \ln x]/2 + D_\mathrm{v}/2\,x \tag{13}$$

where

$$\mathrm{Ei}(\pm 2\,\mathrm{j}\,x) = \mathrm{Ci}(2\,x) \pm \mathrm{j}\,\mathrm{Si}(2\,x)$$
$$\mathrm{Ci}(2\,x) = -\int_{2x}^{\infty}(\cos t/t)\,\mathrm{d}t$$
$$\mathrm{Si}(2\,x) = \int_{0}^{2x}(\sin t)/t\,\mathrm{d}t\,.$$

These functions are tabulated [16], [17], so that a first order approximation of the pressure is determined explicitly. It is interesting to compare the results with those of Benade [1], who derived in a much simpler way the influence of wall damping and only found a logarithmic term. This term is present in eq. (13), and it can be shown to be the largest term in most cases.

The entire derivation is based upon the assumption that D_v and D_t are small. To verify this, the following numerical values are used, as given for instance by Fay [18].

$$\begin{aligned}
\eta &= 1.710 \ (1+0.00288\,T)\,10^{-5}\ \mathrm{Ns/m^2},\\
p_0 &= 101400\ \mathrm{N/m^2},\\
\gamma &= 1.408,\\
\lambda &= 0.0223 \ (1+0.0028\,T)\ \mathrm{J/m\,s\,^\circ C},\\
C_p &= 1010\ \mathrm{J/kg\,^\circ C},\\
T &= 20\ ^\circ\mathrm{C},\\
c &= 340\ \mathrm{m/s}.
\end{aligned}$$

$$\text{Herewith } D_v'(\text{theory}) = 2.9\times 10^{-4}\,k^{\frac{1}{2}}\,\Theta^{-1}$$
$$D_t = 0.46\,D_v'. \tag{14}$$

These numbers refer to the theoretical case of very smooth walls. However, in musical instruments the walls are far from smooth, and they are provided with holes. For a clarinet, radius 7.5 mm, the theoretical quality Q as found from eq. (5) ranges from 30 to 53 for the low register, and from 53 to 75 for the high register. Backus [3] found from experiments 20 to 25 and 40 to 45, respectively. This corresponds with an increase with respect to the theoretical value of D of about 60%. Benade [1] reported augmentations of about 70% for a plastic tube with artificially closed holes.

From these results we estimated, rather arbitrarily, that for conical instruments both D_v' and D_t' are about 60% larger than according to the theory. So:

$$D_v' = 4.5\times 10^{-4}\,k^{0.5}\,\Theta^{-1}. \tag{15}$$

The oboes lowest fundamental is 233 Hz, so $k = 4.31$. From Fig. 1 we find $r_0 = 86$ mm and $\Theta^{-1} \approx 80$ (Russell's Standard Cone [5]). This gives $x_0 = 0.372$ and $D_v' = 0.075$. For higher frequencies D_v' increases with the square root of the frequency. The quantity D_v'/x_0, frequently occurring in the formulas, is 0.20 for the lowest fundamental and decreases with the square root of the frequency. So the first order terms are small, but not very small with respect to unity. An estimate of the accuracy of the solution with only a first order term is attained by looking at the quadratic term. This appears to amount to about 10% of the linear term in D_v. Comparing this with the uncertainty caused by the wall roughness and the presence of holes, we conclude that the first order approximation is sufficiently accurate for a first order study.

By summation, respectively subtraction, of the solutions for p given by eq. (11), two new independent solutions can be formed. They are provided with (complex) constants A and B and are combined to the general solution:

$$p = (A/x)\{\sin x + M + B\ (\cos x + N)\} \tag{16}$$

where

$$\begin{aligned}
M &= D_{vt}\cos x\,(\ln x - \mathrm{Ci}\,2x - \mathrm{Si}\,2x\tan x)/2 + \\
&\quad + D_v \sin x/2x\\
N &= D_{vt}\sin x\,(-\ln x - \mathrm{Ci}\,2x + \mathrm{Si}\,2x\cot x)/2 + \\
&\quad + D_v\cos x/2x\ .
\end{aligned} \tag{17}$$

The particle velocity u is found from eq. (6). When we denote by S the area of the cross section, the acoustic impedance $Z = p/S\,u$ becomes:

$$Z = \frac{\mathrm{j}\,\varrho\,c}{S}\,\frac{x + D_v}{1 - x\,G(B,x)}\,; \tag{18}$$

where

$$G(B,x) = \frac{\cos x + M' + B(-\sin x + N')}{\sin x + M + B(\cos x + N)}\,, \tag{19}$$

$$M' = \mathrm{d}M/\mathrm{d}x,\quad N' = \mathrm{d}N/\mathrm{d}x. \tag{20}$$

The resonance condition for the tube is found after inserting the two boundary conditions into eq. (18).

3. Resonance conditions for a conical woodwind

At the open end with radius a, $r = r_1$, $x = x_1$, $S = S_1$, the radiation impedance is denoted by

$$Z_1 = \mathrm{j}\,\varrho\,c\,(\xi\,k\,a - \mathrm{j}\,\tau)/S_1\,, \tag{21}$$

where ξ and τ are quantities somewhere in between the values for a circular cylinder with infinite flange: $\xi = 8/3\pi \approx 0.85$, $\tau = a^2 k^2/2$, and those for a cylinder without flange $\xi = 0.65$, $\tau = a^2 k^2/4$ [19]. In any case, Z_1 is small with respect to unity. Consequently, the expression which is obtained upon inserting eq. (21) into eq. (18) can be simplified with a good approximation into:

$$\xi\,k\,a - \mathrm{j}\,\tau = 1/G(B, x_1). \tag{22}$$

For open ends like this, a length correction will be introduced onto the preceding tube-piece. Small print will be used for the geometrical length, and capitals for the corrected length, i.e. the length of an imaginary tube with zero terminating impedance resonating in the same frequency. As Z_1 is small, the process is very simple here; to a high accuracy

$$R_1 = r_1 + \xi\,a \ \text{ or } \ X_1 = x_1 + \xi\,k\,a. \tag{23}$$

Herewith B is solved from eq. (22):

$$B = \frac{-\sin X_1 - M_1 + \mathrm{j}\,\tau\cos x_1}{\cos X_1 + N_1 + \mathrm{j}\,\tau\sin x_1}\,. \tag{24}$$

At the throat of the tube ($r = r_0$, $x = x_0$, $S = S_0$) we find a reed with an effective impedance Z_r and, in most cases, an irregularity in the diameter which is described as a cavity V_m in excess over the extended main tube. As this cavity is located in a velocity antinode, it acts as a compliance. The impedance Z_0 looking down the tube must obey the condition

$$1/Z_0 = 1/Z_r - \mathrm{j}\,k\,V_m/\varrho\,c\,. \tag{25}$$

Into this we substitute eq. (18) with $x = x_0$ and $S = S_0$:

$$(\mathrm{j}+\delta)\,\varrho\,c/Z_0'\,S_0 - 1/x_m = -G(B, x_0)\,, \tag{26}$$

where

$$(\mathrm{j}+\delta)\,\varrho\,c/Z_0'\,S_0 = \left(1 + \frac{D_v}{x_0}\right)\frac{\mathrm{j}\,\varrho\,c}{Z_r\,S_0} + D_v\left(\frac{V_m}{S_0\,r_0}\right); \tag{27}$$

$$x_m = x_0/[1 - x_0^2\,V_m/S_0\,r_0]\,. \tag{28}$$

As was found experimentally by Backus [3], the tangent of the phase angle, δ, is mainly due to the reed action and is around unity. Z_0' is real and consists mainly of the real part of the reed impedance. From eqs. (24) and (26), a rather intricate expression for the resonance condition can be found. The meaning of this expression can be clarified, when taking $V_m = \delta = D_v = D_t = 0$. Then:

$$Z_0 = \frac{\mathrm{j}\,\varrho\,c\,x_0/S_0}{1 - x_0\cot(x_0 - X_1)}\,. \tag{29}$$

Two extreme cases can be distinguished. For an open end, Z_0 is very low and the solutions are of the type $(X_1 - x_0) = n\,\pi$, n integer. Because for a reed closure one always finds $|Z_0| > \varrho\,c/S_0$, these solutions do not occur in reality. For very large values of Z_0, to a first approximation for small x_0, $X_1 = n\,\pi$, n integer. This solution is closer to reality, and we use its result for some simplifications, namely $\sin X_1 \approx \sin x_1 \approx 0$ and $\cos X_1 \approx \cos x_1$. The resonance condition as found from eqs. (24) and (26) then takes on the form:

$$(\mathrm{j}+\delta)\,\varrho\,c/Z_0'\,S_0 - 1/x_m = (1+\Phi)\cot(X_1 - x_0)\,, \tag{30}$$

where

$$\Phi = (M_1\sec X_1 - \mathrm{j}\,\tau)\,(\tan x_0 + \cot x_0) + M_0'\sec x_0 - M_0\operatorname{cosec} x_0\,. \tag{31}$$

Using eqs. (8) and (17), Φ may be separated in real and imaginary parts:

$$\left.\begin{aligned} &\Phi = \Phi_1 - \mathrm{j}\,\Phi_2 \\ &\Phi_1 = D_{vt}'(\ln x_1/x_0 - \mathrm{Ci}\,2\,x_1 + \mathrm{Ci}\,2\,x_0) \times \\ &\quad\times(\tan x_0 + \cot x_0)/2 - D_v'\tan x_0/2\,x_0^2 \\ &\Phi_2 = \Phi_1 + \tau(\tan x_0 + \cot x_0)\,. \end{aligned}\right\} \tag{32}$$

Now eq. (30) can be separated into real and imaginary parts too. Elimination of Z_0 between those parts leads to an expression for the resonance frequency:

$$\tan(x_0 - X_1) = x_m(1 + \Phi_1 + \delta\,\Phi_2) \tag{33}$$

or:

$$X_1 = n\,\pi - x_0 - \arctan[x_m(1 + \Phi_1 + \delta\,\Phi_2)]\,. \tag{34}$$

So indeed to a first approximation $X_1 = n\,\pi$, confirming the assumption made above. This means that the oboe oscillates in the same frequency as a cylindrical tube, open at both ends, which has a length equal to the oboe cone from the extrapolated top on.

The imaginary part of eq. (30) delivers the real part of the impedance at the throat:

$$Z_0' = (\varrho\,c/\Phi_2\,S_0)\tan(x_0 - X_1)\,. \tag{35}$$

The deviations from the "ideal" behaviour can be described as length corrections to the ideal length R_1 of the tube to its apex. We may distinguish a mouthpiece cavity correction Δl_m, a wall friction correction Δl_w, and a correction due to complex impedance at the tube entrance, Δl_δ. They are – more or less arbitrarily – defined from:

$$\begin{aligned} k(r_0 + \Delta l_m) &= \arctan x_m\,, \\ k(r_0 + \Delta l_m + \Delta l_w) &= \arctan[x_m(1+\Phi_1)]\,, \\ k(r_0 + \Delta l_m + \Delta l_w + \Delta l_\delta) &= \arctan[x_m(1 + \Phi_1 + \\ &\quad + \delta\,\Phi_2)] \end{aligned}$$

or:

$$\begin{aligned}\Delta l_m &= (r_0/x_0)\arctan x_m - r_0 \approx \\ &\approx (V_m/S_0\,r_0 - 1/3)\,x_0^2\,r_0\,;\end{aligned} \tag{36}$$

$$\Delta l_w = \frac{r_0}{x_0}\arctan\left[\frac{\Phi_1\,x_m}{1 + x_m^2(1+\Phi_1)}\right]; \tag{37}$$

$$\Delta l_\delta = \frac{r_0}{x_0}\arctan\left[\frac{\delta\,\Phi_2\,x_m}{1 + x_m^2(1+\Phi_1)\,(1 + \Phi_1 + \delta\,\Phi_2)}\right] \tag{38}$$

The absolute value of the mouthpiece cavity correction Δl_m remains smaller than $0.02\,r_0$ for $x_0 < 1.5$, when $V_m/S_0\,r_0 \approx 0.35$, which is very plausible from eq. (36). As was calculated before, for the lowest fundamental $x_0 = 0.372$ which means that about two octaves are reasonably pure. According to eq. (32), using eqs. (14) and (15), Φ_1 is calculated and plotted in Fig. 3 as a function of x_0. The corresponding $R_1 = 730 \times 0.372/n\,x_0$ (n = mode number) is indicated on the horizontal scale too. From this, Δl_w is determined with eq. (37), and plotted in

Fig. 4. In Fig. 3, $\Delta l_w/r_0$ is plotted. It can be verified easily that $\tau \ll \Phi_1$, so that $\Phi_2 \approx \Phi_1$ and $\Delta l_\delta \approx \delta\, \Delta l_w$. So for Δl_δ, the same curve may be used as for Δl_w. We remark that both effects apparently have the tendency to lower the pitch.

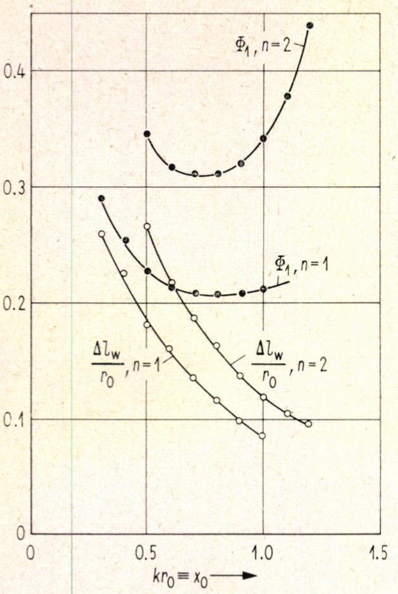

Fig. 3. Function Φ_1 and length correction relative to cone truncation $\Delta l_w/r_0$, both for wall damping, plotted as a function of $k\,r_0$.

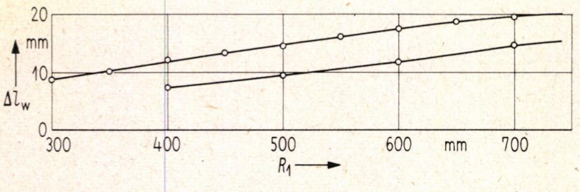

Fig. 4. Length correction due to wall damping Δl_w for first $(n=1)$ and second $(n=2)$ vibrational mode of an oboe, as a function of effective resonating length R_1.

4. Calculation of the hole position

For a hole calculation we used the same method as described earlier [8]. The throat of the tube is assumed to be closed (infinite impedance), and

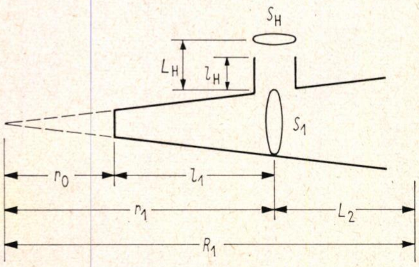

Fig. 5. Explanation of nomenclature for a tube with a single side-hole.

losses are neglected. The tube with a row of opened holes is replaced by one without holes, and resonating in the same frequency. Then a single hole is drilled (see Fig. 5). The terminating impedance Z_1 of the preceding tube-part l_1 is considered to consist of two impedances in parallel, use eq. (29):

$$\frac{1}{Z_1} = \frac{S_H \cot k\, L_H}{\mathrm{j}\,\varrho\, c} + \frac{1 + x_1 \cot k\, L_2}{\mathrm{j}\,\varrho\, c\, x_1/S_1}. \qquad (39)$$

Z_1 is found from eq. (18), in which the boundary condition $Z = \infty$ for $x = x_0$ gives $B = \tan(\xi_0 - x_0)$, where $\xi_0 = \arctan x_0$. So:

$$Z_1 = \frac{\mathrm{j}\,\varrho\, c\, x_1/S_1}{1 - x_1 \cot \xi_1}, \qquad (40)$$

where $\xi_1 = x_1 + \xi_0 - x_0$. Deviations from the assumption $x_0 \ll 1$ have been dealt with before, so we put $\xi_1 = x_1$. Then Z_1 is eliminated from eqs. (39) and (40), and the resonance condition is written in the form:

$$\tan X_1 = 0,$$

where

$$X_1 = x_1 + \operatorname{arc\,cot}[\cot k\, L_2 + (S_H/S_1) \cot k\, L_H]. \quad (41)$$

L_2 and L_H are sufficiently small, so

$$R_1 = r_1 + (S_1\, L_H\, L_2)/(S_H\, L_2 + S_1\, L_H). \quad (42)$$

By substitution of

$$(1+g) = 2^{v/12} \text{ and } L_2 = (1+g) R_1 - r_1,$$

where v = number of semitones of the shift, when the hole opens, we may solve eq. (42) for $R_1 - r_1$ which leads to an expression for the hole position:

$$r_1 = R_1 \left[1 - \frac{g}{2} \left\{ \sqrt{1 + \frac{4\, S_1\, L_H}{g\, S_H\, R_1}} - 1 \right\} \right]. \quad (43)$$

R_1 is the effective length corresponding to the frequency with open hole. This expression is exactly the same as was derived for a cylindrical tube [7], and it is generally valid, for any hole in a row. The effective length of the side hole is the sum of the geometrical length and an end-correction: $L_H = l_H + \Delta l_H$. Recently, LOUDEN published investigations concerning the magnitude of this end-correction [20]. It appeared to be independent of the hole length, but to depend on the ratio of the diameters of hole and main tube, d/D. The end-correction was expressed as $\Delta L_H = \pi\, r^2_H/C_0$ [20], page 32, where r_H is the radius of the hole and C_0 was determined experimentally. The seven experiments of LOUDEN as combined from his Tables on pages 37 and 40 are given in Table I. LOUDEN described the results with

$$C_0 = 0.57\,(d/D) + 5.78\,(d/D)^2. \qquad (44)$$

A serious objection against this expression is that it is dimensionless, ΔL_H becoming an area instead of a length. Therefore, we tried to interpret LOUDEN's measurements on the basis of an assumption that $\Delta l_{\mathrm{H}} = \xi' d$, where $\xi' = \pi d/4 C_0$ and dependent on d/D. From a plot of ξ' versus d/D in Fig. 6 we see that $0.3 < \xi' < 0.8$. The upper limit for a shallow hole can be expected to be 0.85 because the velocity profile has to change twice, so that twice the correction for a normal flanged open end is reasonable. For wider holes, corrections amount to about half as much, indicating that at the inside of the hole none or a small correction appears when d/D is unity. In the oboe mostly $d/D \approx 0.5$ so that our choice is:

$$\Delta l_{\mathrm{H}} = 0.8\, d_{\mathrm{H}}\,. \tag{45}$$

Table I.
Survey of LOUDEN's measurements [20] concerning end-corrections of side-holes.

d cm	d/D	C_0
1.5	0.54	2.00
1.75	0.62	2.21
2.0	0.59	2.45
2.0	0.71	3.55
2.5	0.66	2.91
2.5	0.73	3.62
2.5	0.89	5.03

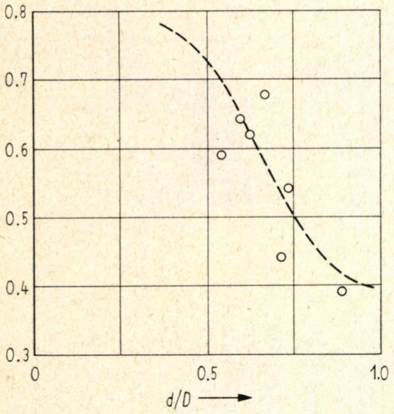

Fig. 6. End-correction for a hole, ξ', expressed in units hole diameter, as a function of the ratio of hole and main-tube diameter, d/D, after measurements of LOUDEN [20].

5. Corrections

As can be seen from Fig. 1, and as has been mentioned before, the diameter of the oboe is far from a perfect cone. The conicity changes several times, and at the end of the staple even with a sudden diameter decrease. These deviations can be accounted for in corrections with respect to one of the cones. The resonance condition for two succeeding cones, the first one closed at the throat and the second one open at the bottom, can be determined by equating impedances at the junction point. The nomenclature is explained in Fig. 7. When using eqs. (29) and (40), the condition $Z_{11} = Z_1$ leads to:

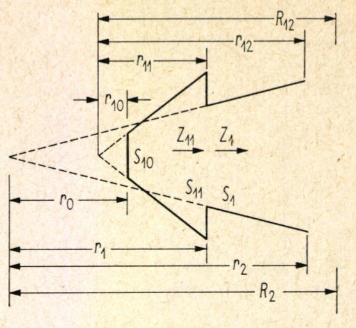

Fig. 7. Two succeeding cones with different conicity and diameter. Double subscripts refer to the first, and single subscripts to the second cone.

$$\frac{x_{11}/S_{11}}{1 - x_{11}\cot\xi_{11}} = \frac{x_1/S_1}{1 - x_1\cot(x_1 - X_2)}\,. \tag{46}$$

We first apply this to the situation at the mouthpiece. As was argued before, a positive mouthpiece cavity is a necessity for obtaining pure overtones. From Fig. 1, however, we see that there is no cavity at the tip of the reed. Instead of that there is a widening in the bore at a place somewhat further down the tube. The diameter follows more or less the course of a cone deviating markedly from that of the main tube. The dimensions of this horn, together with those of the extrapolated main tube, are listed in Table II. The volumes of the cone approximation and the extrapolated main tube can be calculated, and they are both about 0.52 cm³. A measurement of the volume of staple and reed with a calibrated injection needle yielded 0.55 cm³ to 0.61 cm³, depending on the lip pressure on the reeds. This means that a mouthpiece cavity, in the sense as discussed before, hardly exists. The same effect for overtones, however, is performed by the conical irregularity. To show this, eq. (46) is written as follows:

Table II.
Survey of dimensions of the two cones at the top of the oboe.

	diameter mm	distance mm
main tube	$d_0 = 1.11$ $d_1 = 2.0$	$r_0 = 86$ $r_1 = 154$
staple and reed	$d_{01} = 0.87$ $d_{11} = 2.4$	$r_{01} = 33$ $r_{11} = 91$

$$\tan(x_1 - X_2) = \frac{x_{11} S_1 x_1 \tan\xi_{11}}{(S_1 x_{11} - S_{11} x_1)\tan\xi_{11} + S_{11} x_1 x_{11}}\,. \tag{47}$$

A power series development of than ξ_{11} up to third power terms gives:

$$\tan \xi_{11} = x_{11} + (x_{11}^3 - x_{10}^3)/3 .$$

As $x_{11} \approx 3\, x_{10}$, the third power of the latter is neglected. Then eq. (47) becomes:

$$\tan(x_1 - X_2) = \frac{x_1(1 + x_{11}{}^2/3)}{1 + x_{11}{}^2/3 - S_{11}\, x_1\, x_{11}/3\, S_1} . \quad (48)$$

We find, accurate to quadratic terms,

$$X_2 = n\,\pi + x_1 - \operatorname{arc\,tan} \frac{x_1}{1 - (S_{11}\, x_{11}/3\, S_1\, x_1)\, x_1{}^2} . \quad (49)$$

Using numerical values from Table II, we find that the factor in front of $x_1{}^2$ is about 0.3. Eq. (49) is similar to eq. (34) and for the same reason the coefficient of the quadratic term here should be about 0.35 for a best approximation of the resonance condition $X_2 = n\,\pi$. We conclude that the sense of the peculiar cone at the top of the instrument is mainly purity of overtones.

At the bottom of the tube, a very rapidly expanding conical horn is found. We use the same nomenclature as before. The horn is very short, so $\tan(x_1 - X_2) = x_1 - X_2$, x_1/x_{11} is neglected and $S_1 = S_{11}$. Now from eq. (47):

$$\tan \xi_{11} = -x_1(1 - x_1/X_2) . \quad (50)$$

This means a length correction of $r_1(1 - r_1/R_2)$ onto the preceding tube, or a correction to the tube the horn including, of:

$$\Delta l_{\mathrm{h}} = r_1(2 - r_1/R_2) - r_2 . \quad (51)$$

The lower part of the oboe is a slightly sharper cone than the upper part, and a correction will be applied to the extrapolated upper part. We use eq. (46), with $S_1 = S_{11}$, replace $(x_1 - X_2)$ by $(x_{11} - X_{12})$ and take $\xi_{11} = x_{11}$. Then:

$$-1/x_{11} + 1/x_1 + \cot x_{11} = \cot(x_{11} - X_{12}) . \quad (52)$$

This can be reshaped in functions of $\tan x_{11}$ and $\tan X_{12}$ so as to lead to the condition $\sin(X_{12} + k\,\Delta l_{\mathrm{c}}) = 0$, where approximately

$$\Delta l_{\mathrm{c}} = \frac{r_1 - r_{11}}{x_1\, x_{11}/\sin^2 x_{11} - (x_1 - x_{11}) \cot x_{11}} .$$

When $(r_1 - r_{11})\,(\sin 2\, x_{11})/(2\, r_1\, x_{11}) \ll 1$:

$$\Delta l_{\mathrm{c}} = \frac{r_{11}}{r_1}\,(r_1 - r_{11}) \left(\frac{\sin x_{11}}{x_{11}}\right)^2 . \quad (53)$$

This formula can be used for the three oboes of Fig. 2. After extrapolating the straight lines to the line $d = 0$, we obtain for the Mönnich $r_1 = 337$ mm, $r_{11} = 320$ mm, for the Lorée $r_1 = 328$ mm, $r_{11} = 304$ mm, for the Marigaux $r_1 = 456$ mm, $r_{11} = 418$ mm.

In Fig. 8 the corrections as calculated with eq. (53) are given for the three oboes. It can be seen that the influence for the second vibrational mode is

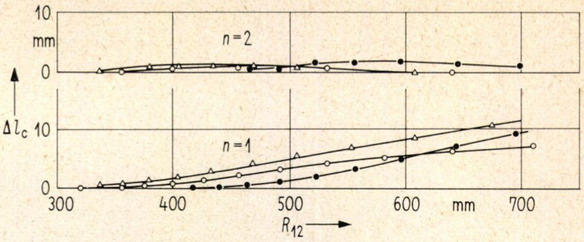

Fig. 8. Length correction Δl_{c} due to the conicity change in the middle for the three oboes of Fig. 2. The correction is plotted against the effective resonating length R_{12} for two vibrational modes $n = 1$ and $n = 2$.
● Marigaux,
△ Lorée,
○ Mönnich.

small, but that the ground-mode is flattened some tenths of a semitone.

The influence of the row of closed side-holes in a conical tube is accounted for in an analogous way as was done for a cylindrical tube in a previous paper [8]. From eqs. (1), (8) and (13) of that paper we derive, changing the sign of Z_1 because

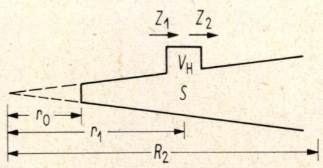

Fig. 9. Nomenclature as used with the calculation of the influence of a closed side-hole on the resonance frequency.

here (cf. Fig. 9) both impedances are taken looking down the tube,

$$Z_1 = Z_2 - \mathrm{j}\, k\, V_{\mathrm{H}}(Z_1\, Z_2/\varrho\, c + \varepsilon\, \varrho\, c/S^2) . \quad (54)$$

V_{H} denotes the volume of the side hole, ε is a factor dependent on the shape:

$$\varepsilon = (2/\pi) \operatorname{arc\,tan}(2\, b/13\, a) , \quad (55)$$

where b/a is the ratio width-depth of the hole. Impedance Z_1 is given by eq. (40) with $\xi_1 = x_1$, and Z_2 is found from eq. (29) by replacing x_0 and X_1 by x_1 and X_2. Proceeding in the same way as before, we obtain a resonance condition $\tan k(R_3 + \Delta l) = 0$, where for small Δl:

$$\Delta l = (V_{\mathrm{H}}/S)\, [\sin^2 x_1 - \varepsilon (\cos x_1 - x_1{}^{-1} \sin x_1)^2] . \quad (56)$$

For a row of side-holes for which $\Sigma(V_{\mathrm{H}}/S)$ is proportional to R, the total correction can be found by integration $(x = k\,r)$

$$\Delta l_{\mathrm{V}} = \int_{r_3}^{R_1} \frac{\Sigma(V_{\mathrm{H}}/S)}{R_1 - r_3} \left[\sin^2 x - \varepsilon \left(\cos x - \frac{\sin x}{x} \right)^2 \right] \mathrm{d}r\,.$$

When using the approximate relation $\sin X_1 = 0$, we find, n integer,

$$\Delta l_{\mathrm{V}} = \sum \frac{V_{\mathrm{H}}}{2\,S} \left[1 - \varepsilon - (1+\varepsilon) \frac{\sin\varphi}{\varphi} - \frac{4\,\varepsilon(1-\cos\varphi)}{\varphi(2\,n\,\pi - \varphi)} \right]$$
$$\varphi = 2\,n\,\pi(1 - r_3/R_1)\,. \qquad (57)$$

Finally some miscellaneous irregularities in the bore are dealt with by using eq. (56). The volume of the constriction in the top divided by the local area for respectively the Mönnich, Lorée and Marigaux is about $V_{\mathrm{H}}/S = 0.6$ mm, 3.0 mm and 2.6 mm; r_1 is assumed 200 mm. The corrections are largest for the Lorée, and these are plotted in Fig. 10. For

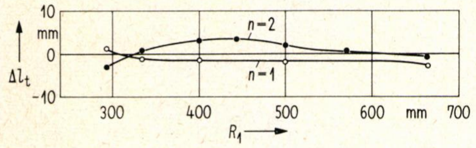

Fig. 10. Length correction Δl_{t} due to the constriction at the top of the Lorée oboe, plotted against the effective resonating length R_1 for two vibrational modes $n = 1$ and $n = 2$. For the Marigaux and the Mönnich this correction is 0.87, respectively 0.2 times, as much.

the other instruments, they are smaller to the ratio of V_{H}/S. The constriction and expansion in the middle of the Mönnich can be assumed to have a negligible influence, because $V_{\mathrm{H}}/\mathrm{S} = 0.9$ mm.

On comparing Figs. 8 and 10 we conclude that to some extent the effects of the sharpening conicity and the constriction just below the staple compensate each other. If there would be no other reason for their existence, it seems they could be left out both. According to Bate [6], page 83, modern oboes show fewer deviations from conicity than older ones, and one might suggest that this has something to do with it.

6. Application

Russell's extensive measurements on oboes [5] cannot be used here, because these mainly concern the diameter and only occasionally hole diameters. For comparison with reality, we investigated the Mönnich oboe, conservatory system, in the possession and played by de Bruijn. Its lowest fundamental is 233 Hz (B-flat). The specifications pertinent to the holes are listed in Table III. The tones sounding with closed and opened hole are given. Holes 1 to 11 are located in the upper joint, 12 to 21 in the lower branch, hole 22 in the bell. The lowest note sounding with all holes closed is listed at the bottom of the Table. Holes 1, 2 and 5 serve as an aid for overblowing and have no specific tuning function. Some holes serve two purposes, and are consequently calculated for both. Due to the intricacies of the mechanism, in certain situations two keys move together. So the notes wich belong with a given hole are flat or sharp, as indicated with a minus or plus sign. The deviations are perceptible, but small, and it suffices to apply the formula for the full shift. The diameter of a hole, d_{H}, is not always defined uniquely. Some holes, indicated in the second column with "u" show so-called undercutting, i.e. the hole is conically widened from the inside. Also, the length of the hole l_{H}, is not defined. The outer end may be curved. Mostly, however, some wall material is removed so as to obtain a circular end for a better fit onto fingers and pads. The inside end is always curved, especially in case of undercutting. We chose to specify the shortest length of the hole. l_{g} is the distance measured from the reed tip to the centre of the hole. d_1 was found from Figs. 1 and 2.

The calculation of the hole position to the apex of the horn, r_1, was performed with eq. (43). The results are given in Table III. For R_1 was used $170/f$ where frequencies f were chosen according to equal temperament.

For the lowest note, the correction in the bell is introduced, see eq. (51), with r_1 = 70 mm, r_2 = 160 mm and $R_2 = 170$ mm, $\Delta l_{\mathrm{h}} = -49$ mm. Hole 22 is located in the horn and we applied a correction for this, using eq. (56); it simplifies for this case into $\Delta l_{\mathrm{h}} = -\Delta V/S_0$; and we find $\Delta l_{\mathrm{h}} = -4$ mm. The differences between calculated and measured posi-

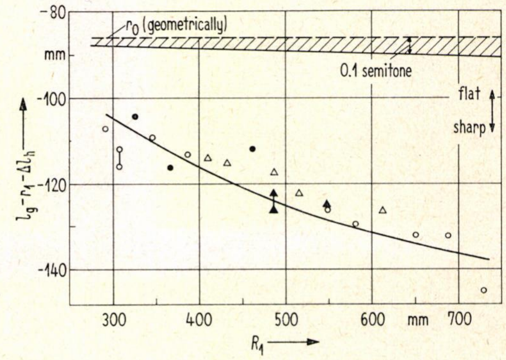

Fig. 11. Difference between real and calculated hole position including the bell-horn correction, as a function of effective length R_1. A shaded area indicates a 0.1 semitone shift. Triangular symbols refer to holes with undercutting, filled circles refer to holes for which the frequency shift differs slightly from that assumed in the calculations.

Table III.
Survey of hole specifications.

hole	key (k) hole (o) under-cutting (u)	tone with hole		R_1	d_1	d_H	l_H	l_g	calculated	
		closed	open	mm	mm	mm	mm	mm	Δl_h	r_1
1	k					0.3		73		
2	k					0.3		129		
3	k	c^1	d^1	289	6.7	3.1	6	158		265
4a	k	c^1	$c^1\#$	307	6.7	3.0	6	170		286
4b		b	$c^1\#$	307						282
5	k				7.2	4.2	6	182		
6	k	b^+	c^1	325	7.6	3.6	6	201		305
7	o	a	b	344	7.9	4.8	8.5	213		312
8	k	a	$a\#^+$	365	8.3	4.2	6	230		346
9	k	a	$a\#^+$	365	8.3	4.2	6	230		346
10	o	g	a	386	8.7	4.7	8.5	246		364
11	k, u	g	$g\#$	409	9.5	5.0	6	275		389
12	k, u	$f\#$	g	434	10.2	7.0	5.5	304		419
13	k	f^+	$f\#$	460	10.6	4.6	6	321		433
14a	o, u	e	f^+	486	11.0	5.6	6	337		463
14b		$d\#$	f	486						459
15	k, u	e	f	486	11.3	7.3	6	351		468
16	o, u	d	e	516	11.7	6.7	8.5	365		492
17	k	d	$d\#$	546	12.6	7.6	5.5	401		526
18	k, u	d	$d\#$	546	12.6	8.5	4.8	403		529
19	k	c	d	579	13.3	9.7	4.5	431		560
20	k, u	c	$c\#$	613	14.0	7.6	5.4	463		589
21	k	B	c	650	14.8	9.8	4.5	499		631
22	k	$A\#$	B	688	16.5	10.8	4.5	539	−4	667
—		$A\#$		730		—		634	−49	730

tions, corrected for the horn in the bell, $l_g - r_1 + \Delta l_h$, are plotted in Fig. 11 as a function of R_1. Points referring to double functions of the same hole are connected with a vertical line. A scatter is observed. Some of this is due to undercutting (triangular symbols). If we should correct for this, triangulars and black circles are expected to move downwards some mm in the diagram. This would reduce the scatter somewhat. Taking this effect into account, a smooth curve was drawn through the separate points. The remaining scatter is ± 0.1 semitone (0.1 semitone is about $0.006\,R_1$), which is often considered as admissible for practice [4], as it can be corrected by lip pressure.

The smooth curve was used as a basis to which the corrections were applied as derived in the preceding paragraphs. To determine the closed side-hole correction, $\Sigma\, V_H/S$ was plotted in Fig. 12 as a function of R_1. The mean value of b/a was found to be 0.7, so with eq. (55), $\varepsilon = 0.07$. The correction Δl_V calculated with eq. (57) taking $r_3 = 275$ mm is plotted in Fig. 12 for two vibrational modes, $n = 1$ and $n = 2$. The final deviation from real and calculated position

$$\Delta l_{tot} = l_g - r_1 + \Delta l_h + r_0 + \Delta l_c + \Delta l_V + \Delta l_w + \Delta l_\delta \tag{57}$$

is plotted in Fig. 13 for $n = 1$ and $n = 2$ and for $\delta = 0$

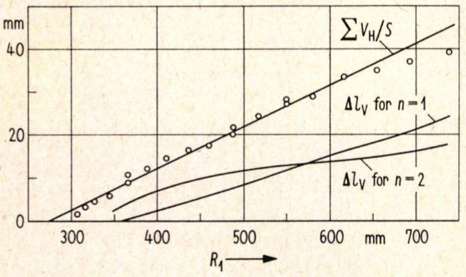

Fig. 12. Sum of closed side-hole volume divided by main tube area, $\Sigma(V_H/S)$, for the row of closed side-holes, Δl_V, as a function of the effective resonating length R_1. Curves are given for vibrational modes $n = 1$ and $n = 2$.

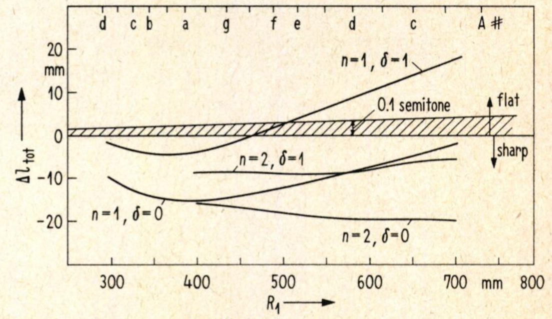

Fig. 13. Difference Δl_{tot} between real and calculated hole positions, including the most important corrections, is plotted versus effective length R_1 for first and second mode, $n = 1$ and $n = 2$ and for tangent of the phase angle at the reed $\delta = 0$ and $\delta = 1$. Sounding tones in the first mode are indicated. A shaded area indicates a 0.1 semitone shift.

and $\delta = 1$. In the diagram, a shaded area indicates a shift of 0.1 semitone. A first impression, when observing Fig. 10 is that the lowest group of notes is flat. This corresponds with the findings of YOUNG [4], who concluded that in particular an inexperienced player (who is supposed not to correct errors automatically with his lips) tends to intonate low notes up to half a semitone flat, which is about the amount theoretically found here. It is a known fact that oboe troubles always concern a flatness in low notes, contrary to the clarinet where low notes tend to be sharp. It is striking that clarinet and oboe calculations both predict deviations in the same directions as where practice finds these [8], [9]. On the other hand, we may imagine that there might be a reason for this flattening of a whole region, because this effect could easily have been compensated for by widening the lower part of the oboe instead of narrowing it. From Fig. 8 it can be seen that such a measure will have the right effect. The sharpening conicity, however, seems to be a feature of an oboe; most of the oboes investigated by RUSSELL [20] showed it. We can imagine that it has some function, for instance tone colour, tuning of high notes, ease of embouchure.

7. Conclusions

The oboe bore approximates to a truncated cone and is effectively closed at the top and open at the bottom. Its resonance frequency is about equal to that of an open-open cylindrical tube with a length equal to the cone extrapolated to its apex.

The wave equation in a conical tube, when solved up to first order terms, yields corrections due to radiation, wall damping and complex reed impedance. Other corrections due to deviations of the ideal cone, and the closed side-holes, are introduced. The effect of all corrections together is a flattening of the resonance frequency by one half to a whole tone.

The position of a hole can be predicted with respect to neighbouring holes within 0.1 to 0.2 of a semitone. Predicting the hole position without knowledge of its neighbours is less accurate, but still reasonable, except that the theory calculates the lowest notes up to about half a semitone flat.

The reported investigations may help to indicate an individual impure note, and permit its correction. They may also assist in efforts to change the bore for retuning part of a register of a conical woodwind.

Acknowledgements

We thank Ir. D. W. VAN WULFFTEN PALTHE of the Technological University, Delft, for his valuable suggestions concerning the mathematics of this paper, and Mr. E. J. SPELBERG and Mr. F. MINDERAA, principal oboes of the Hague Philharmonic Orchestra, for kindly submitting their professional instruments to a scientific test.

(Received June 6th, 1966.)

References

[1] BENADE, A. H., J. Acoust. Soc. Amer. **31** [1959], 137.
[2] BENADE, A. H., J. Acoust. Soc. Amer. **32** [1960], 1591.
[3] BACKUS, J., J. Acoust. Soc. Amer. **35** [1963], 305.
[4] YOUNG, R. W., Gravesaner Blätter, H. 7/8 [1957], 87; H. 9 [1957], 111; H. 11/12 [1958], 182.
[5] RUSSELL, M. E., The oboe, unpublished dissertation, Iowa State Teachers College, Ceder Falls, Iowa, 1953.
[6] BATE, P., The oboe, 2nd edition. Ernest Benn Ltd. London 1962.
[7] NEDERVEEN, C. J. and v. WULFFTEN PALTHE, D. W., Acustica **13** [1963], 65.
[8] NEDERVEEN, C. J., Acustica **14** [1964], 227.
[9] YOUNG, R. W., A decade of Musical Acoustics, Invited paper at the Fourth International Congress on Acoustics, Copenhagen 1962.
[10] MORSE, P. M., Vibrations and Sound, 2nd edition. McGraw-Hill Book Co., New York 1948.
[11] REDFIELD, J., J. Acoust. Soc. Amer. **6** [1934], 34.
[12] RAYLEIGH, J. W. S., Theory of Sound, Vol. II. Dover, New York 1945, p. 319.
[13] CRANDALL, I. B., Theory of Vibrating Systems and Sound. MacMillan & Co., London 1927, Appendix A.
[14] ZWIKKER, C. and KOSTEN, C. W., Sound Absorbing Materials. Elsevier, Amsterdam 1949, p. 25.
[15] JAHNKE, E. and EMDE, F., Tables of functions, Dover, New York 1945, p. 137.
[16] JAHNKE, E. and EMDE, F., Tables of functions, Dover, New York 1945, p. 1.
[17] Handbook of Chemistry and Physics. Chemical Rubber Publishing Cy., Cleveland (Ohio) 1963, p. 291.
[18] FAY, R. D., J. Acoust. Soc. Amer. **12** [1940], 62.
[19] KINSLER, L. E. and FREY, A. R., Fundamentals of Acoustics. J. Wiley & Sons, New York 1950, p. 213.
[20] LOUDEN, M. M. E., Dissertation, TH Braunschweig, 1964.