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# EXPLAINING AND RECTIFYING AHP REVERSALS

Rank reversal can occur in the Analytic Hierarchy Process when alternatives are added or deleted. Recently, it was found that removing a non-discriminating criterion could cause the same phenomenon. This paper offers an explanation of these rank reversals and shows how they can be avoided through a link between the normalization and weighting processes.

## Introduction

It has long been known with the Analytic Hierarchy Process (AHP) that addition or deletion of a copy of an existing alternative can cause the rank of other alternatives to reverse (Belton and Gear, 1983). Recently, Finan and Hurley (2002) discovered that removing a non-discriminating criterion from a multilevel AHP hierarchy causes the same phenomenon. This makes AHP appear deficient, since such removals should have no effect upon the final rank order. It has also been known that addition or removal of alternatives not being copies of existing alternatives can cause ranks to reverse (Saaty, 2000).

A non-discriminating criterion exists when a decision-maker is indifferent among the alternatives when they are compared on that criterion. Since non-discriminating criteria do not differentiate between the alternatives, it is presumably safe to eliminate them from further consideration. Similarly, the addition or removal of an alternative that is independent of other alternatives should have no effect on the final ranking of alternatives.

We take as a starting point a previously published paper by Finan & Hurley (2002). In that paper, they denote a non-discriminating criterion as a “wash criterion” and investigate how the final rank order of the alternatives is affected by removing such a criterion from an AHP hierarchy. They differentiate between single-level hierarchies that have only one level of criteria below the goal, and multilevel hierarchies, that have two or more criteria levels. They show that, assuming a perfectly consistent decision-maker, the final rank order of the alternatives is never affected by removing a non-discriminating criterion from a single-level hierarchy. But using a simple example of a two-level hierarchy, they show that in multilevel hierarchies, leaving out a non-discriminating criterion can reverse the final rank order. This is an interesting observation, since most literature on rank reversal in AHP relates to the addition or deletion of alternatives, not criteria (ref. Saaty, 2000 and Belton & Stewart, 2002 ).

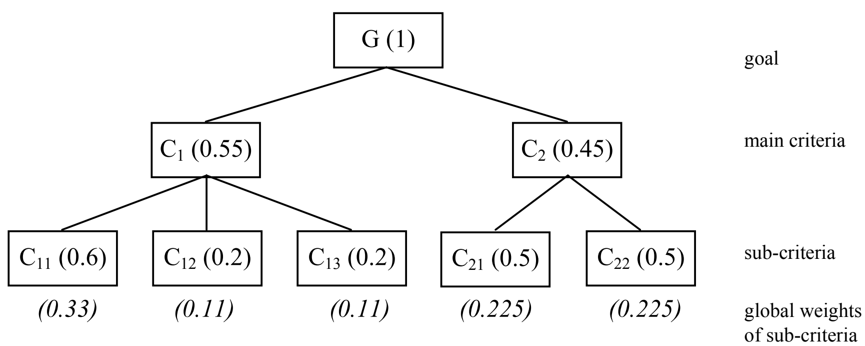
Finan & Hurley conclude that, since any hierarchy with multiple levels of criteria can be modelled as a hierarchy with a single level of criteria, the methods of synthesizing a multilevel AHP hierarchy must be incorrect. Their comments add to the challenge of AHP methodology, but they provide no explanation for the defect nor do they offer any type of resolution. In this paper we do both.

We use the example provided by Finan and Hurley to show why rank reversals occur. We do this for both the addition/removal of alternatives and for the synthesis of a two-level hierarchy where a non-discriminating criterion has been removed. We suggest that there is a necessary link between the normalization and weighting processes and that Finan & Hurley (and others) have failed to understand the meaning of the unit of measure in a multiple level hierarchy. We show that the rank reversal problem is avoided by freezing the unit of measure or by proper adjustment of the appropriate weights if the unit of measure changes. We also prove that rank reversal will never occur when removing a non-discriminating criterion from a single-level hierarchy, even if the decision-maker is not perfectly consistent. We end by discussing whether or not one should remove non-discriminating criteria in the first place.

## The Finan and Hurley Example

For our calculations and illustrations, we adopt Finan & Hurley's example of a hierarchy with two levels of criteria below the goal G: two main criteria ($C_1$, $C_2$) on the first level and three sub-criteria of $C_1$ ($C_{11}$, $C_{12}$, $C_{13}$) and two sub-criteria of $C_2$ ($C_{21}$, $C_{22}$) on the second. Their respective local weights are shown in the boxes. The local priorities of two alternatives $A_1$ and $A_2$ are shown as well. Below sub-criteria, the global weights are in italics.

**Figure 1: Example of a Hierarchy with One Non-discriminating Sub-criterion**




| | | | | | | |
|---|---|---|---|---|---|---|
| $A_1$: | 0.5 | 0.8 | 0.4 | 0.2 | 0.6 | local priorities |
| $A_2$: | 0.5 | 0.2 | 0.6 | 0.8 | 0.4 | of alternatives |

First, we compute the initial composite priorities of the alternatives as follows, using additive synthesis:

$$A_1 = \{0.5*0.6+0.8*0.2+0.4*0.2\}*0.55 + \{0.2*0.5+0.6*0.5\}*0.45 = 0.477 \qquad (1)$$
$$A_2 = \{0.5*0.6+0.2*0.2+0.6*0.2\}*0.55 + \{0.8*0.5+0.4*0.5\}*0.45 = 0.523$$

From this follows that $A_2$ is 0.523/0.477=1.096 times preferred to $A_1$.

### 1. Example of Rank Reversal with Addition of an Alternative

Given that the above example contains only two alternatives, the deletion of an alternative cannot be considered for the rank reversal situation. But if we add a third alternative, the local weights of the alternatives could be as follows:

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{21}$ | $C_{22}$ |
|---|---|---|---|---|---|
| $A_1$: | 0.333 | 0.72 | 0.12 | 0.1 | 0.54 |
| $A_2$: | 0.333 | 0.18 | 0.18 | 0.4 | 0.36 |
| $A_3$: | 0.333 | 0.1 | 0.7 | 0.5 | 0.1 |

In AHP fashion, these local priorities are normalized to sum to one. Although $A_1$ and $A_2$ have new local values, they have not changed and they maintain their original relative ratios. The new alternative that was added, $A_3$, is relevant to the problem, because it dominates on two of the criteria, $C_{13}$ and $C_{21}$. The composite priorities of the alternatives will now be:

$$A_1 = \{0.333*0.6+0.72*0.2+0.12*0.2\}*0.55 + \{0.1*0.5+0.54*0.5\}*0.45 = 0.346 \quad (2)$$
$$A_2 = \{0.333*0.6+0.18*0.2+0.18*0.2\}*0.55 + \{0.4*0.5+0.36*0.5\}*0.45 = 0.321$$
$$A_3 = \{0.333*0.6+0.1 \ *0.2+0.7 \ *0.2\}*0.55 + \{0.5*0.5+0.1*0.5\} * 0.45 = 0.333$$

From this follows that $A_1$ is 0.346/0.321=1.078 times preferred to $A_2$. Rank reversal has occurred even though nothing changed between $A_1$ and $A_2$. Finan and Hurley did not consider this type of reversal, but it is the standard type well known in AHP literature when alternatives are added or deleted (Saaty, 2000; Belton & Stewart, 2002).

**2. Example of Rank Reversal with Removal of a Non-discriminating Criterion**

The reversals that Finan and Hurley uncovered were those caused by the removal of a non-discriminating criterion. Notice in Figure 1 that the alternatives under $C_{11}$ are equally attractive on that sub-criterion -- $C_{11}$ is a non-discriminating or wash criterion. If $C_{11}$ is removed from the hierarchy and $C_{12}$ and $C_{13}$ are re-normalized to the unit sum, $C_{12}$ and $C_{13}$ will get higher weights: 0.5 each. The new composite priorities of the alternatives with $C_{11}$ removed are as follows (these are Finan and Hurley's findings):

$$A_1 = \{0.8*0.5+0.4*0.5\}*0.55 + \{0.2*0.5+0.6*0.5\}*0.45 = 0.51 \quad (3)$$
$$A_2 = \{0.2*0.5+0.6*0.5\}*0.55 + \{0.8*0.5+0.4*0.5\}*0.45 = 0.49$$

Now, $A_1$ is 0.51/0.49=1.041 times preferred to $A_2$; their ranks are reversed compared with the base results above. This result is similar to the addition or removal of an alternative, yet no such change was made.

**3. Example of No Rank Reversal with Removal of a non-discriminating Criterion from a Single-level Hierarchy**

In AHP synthesis, it is possible to first compute global weights of the sub-criteria by successive multiplication of the local criteria weights and then multiply the local priorities of alternatives by those global weights. This boils down to creating a single-level hierarchy where only the lowest level of criteria (the former sub-criteria in our example) is shown with its global weights. In such a single-level hierarchy there would be no difference between local and global weights of the criteria. These global weights of our example are shown in italics in Figure 1.

Taking $C_{11}$ into account in this single level hierarchy, the composite priorities are:

$$A_1 = 0.5*0.33 + 0.8*0.11 + 0.4*0.11 + 0.2*0.225 + 0.6*0.225 = 0.477 \quad (4)$$
$$A_2 = 0.5*0.33 + 0.2*0.11 + 0.6*0.11 + 0.8*0.225 + 0.4*0.225 = 0.523$$

The results are of course identical to (1). $A_2$ is 0.523/0.477=1.096 times preferred to $A_1$.

Next, we remove non-discriminating $C_{11}$ and compute new composite priorities by simply subtracting 0.5*0.33=0.165 from the previous ones. Thus:

$$A_1 = 0.477 - 0.165 = 0.312 \qquad (5)$$
$$A_2 = 0.523 - 0.165 = 0.358.$$

The new ratio is 0.358/0.312=1.147, thereby heightening the difference between the two alternatives and making $A_2$ more pronounced as the best alternative. We observe no reversal.

We could have first re-normalized the remaining global weights and then computed the weighted sums, as follows:

$$A_1 = 0.8*0.1642 + 0.4*0.1642 + 0.2*0.3358 + 0.6*0.3358 = 0.4657 \qquad (6)$$
$$A_2 = 0.2*0.1642 + 0.6*0.1642 + 0.8*0.3358 + 0.4*0.3358 = 0.5343$$

The composite priorities are of course higher owing to the re-normalization (which heightens the weights), but their rank order is preserved and A2 is still 1.147 more preferred than A1.

We shall now show that in a single-level hierarchy rank reversal will never occur when removing a non-discriminating criterion, even if the decision-maker is not perfectly consistent[1].

Suppose that we have a set $J = \{0, 1, ..., n\}$ of $n$+1 criteria in an AHP hierarchy with one level of criteria below the goal. 0 indexes the non-discriminating criterion. The reduced set is denoted by $\underline{J} = \{1, ..., n\}$. We have criteria weights $c_j$ ($j$=0, ..., $n$) for $J$, with $\sum_j c_j = 1$, and $\underline{c}_j$ ($j$=1, ..., $n$), with $\sum_j \underline{c}_j = 1$ for the reduced set $\underline{J}$. Assuming that we already know the values of $c_j$, with all $c_j$<1, then it is reasonable to conclude that $c_i / c_j = \underline{c}_i / \underline{c}_j$ for $i, j \in \{1, ..., n\}$. Define a constant $\theta$ such that $\underline{c}_i / c_i = \underline{c}_j / c_j = \theta$ for $i, j \in \{1, ..., n\}$. From $\sum_{j=1}^{n} \underline{c}_j = 1 = \theta \sum_{j=1}^{n} c_j = \theta(1 - c_0)$, it follows therefore that $\theta = 1/(1 - c_0)$, and therefore $\underline{c}_j = c_j / (1 - c_0)$ for $j \in \{1, ..., n\}$.

Let the local priority of an alternative $x$ ($x$=1, ..., $m$) on a criterion $i$ be denoted by $u_{xi}$ and its composite priority for the set $J$ be denoted by $w_x$ and for the reduced set $\underline{J}$ by $\underline{w}_x$. In particular, we have $u_{x0} = 1/m$ for all $x \in \{1, ..., m\}$ since the criterion indexed by 0 is non-discriminate. Using additive synthesis, we compute the following difference between the composite priorities of alternatives $x$ and $y$ for the full set $J$ (see also Finan & Hurley, 2002):

$$\begin{aligned} w_x - w_y &= \sum_{k=0}^{n} c_k u_{xk} - \sum_{k=0}^{n} c_k u_{yk} \\ &= c_0/m + \sum_{k=1}^{n} (1-c_0)\underline{c}_k u_{xk} - c_0/m - \sum_{k=1}^{n} (1-c_0)\underline{c}_k u_{yk} \\ &= (1-c_0)\ \{ \sum_{k=1}^{n} \underline{c}_k u_{xk} - \sum_{k=1}^{n} \underline{c}_k u_{yk} \} \\ &= (1-c_0)\ \{\underline{w}_x - \underline{w}_y\} \end{aligned}$$

[1] We are indebted to an anonymous researcher for bringing this to our attention.

Since $(1 - c_0) > 0$, the signs of $(w_x - w_y)$ and $(\underline{w}_x - \underline{w}_y)$ are the same, and, therefore, the rank order is not affected by removing the non-discriminating criterion. Note that we have never used the assumption of a perfectly consistent decision-maker. The above can of course be extended to removal of more than one non-discriminating criterion.

### Avoiding Reversals on Addition or Deletion of Alternatives.

Normalization in AHP methodologies, whether local or global priorities, is generally to a unit sum total. Through successive downward rescaling of local priorities to get global weights, the unit sum of the local weights are converted into a portion of the unit sum of the entire hierarchy. Notice in Figure 1 that this unit of the total hierarchy is found at the topmost node, the Goal, and that the sum of the global priorities on any level below the Goal equals that unit. This implies that the unit for the whole hierarchy is the topmost goal and that all partial values below, alternatives and criteria alike, contribute to that unity. Subsequent synthesis to get the total contribution of each alternative results in the composite alternative values equalling unity (e.g. 0.477 + 0.523 = 1 in Figure 1 and (1)).

### The Source of the Reversal Problem

What is rarely recognized with unit sum normalization is that any different re-normalization produces a new unit of measure. To realize this, take into account a single criterion situation where it is well known that rank reversal never occurs (Saaty, 1990). Let the set $\{Xj \mid j= 1, 2, \ldots, m\}$ be ratio values of a single criterion. The local priority weights of an alternative x (x=1, …, m) will be $u_j = X_j / \Sigma_{j=1}^{m} X_j$. If we add alternative ratio value $X_{m+1}$ to the choice set and re-normalize, the new local priority for an alternative will be $u_j = X_j / (\Sigma_{j=1}^{m} X_j + X_{m+1})$. With $X_{m+1}$ added, the priority unit in which each $u_j$ is measured will be different, since $X_j / \Sigma_{j=1}^{m} X_j \neq X_j / (\Sigma_{j=1}^{m} X_j + X_{m+1})$. Although ratios are maintained for this single criterion situation, the renormalization causes the unit of measure to change.

For example, consider $C_{21}$ of Figure 1 to be the single criterion that represents the totality of the problem. $C_{21}$ would take the unit value of 1 and $A_1$ and $A_2$ would take the relative ratio values of 0.2 and 0.8, respectively. The sum of two alternatives comprises the unit. Next, consider the addition of $A_3$ that is 20% better than $A_2$ on $C_{21}$. If we rescale and re-normalize the ratio values to make the single criterion sum to one, then the new priorities for the alternatives are $A_1 = 0.1$, $A_2 = 0.4$, and $A_3 = 0.5$. We have a new unit of measure, because $A_1$ and $A_2$ no longer equal their original values. They are one-half of their former selves with the same ranks and ratios.

So, what is the implication of this when $A_3$ is added to the multiple criteria hierarchy of Figure 1? The addition and re-normalization with $A_3$ at the bottom level of the hierarchy causes all local priorities below criteria to be expressed in a new unit of measure (original local priorities of alternatives take on new values). Also, the addition of $A_3$ brings more influence to the total hierarchy – in effect, the overall unit of the topmost goal no longer measures the criteria contributions of $A_1$ and $A_2$ but also $A_3$. Since the former criteria weights no longer represent the correct relative amounts that each alternative brings to the goal, using them distorts the global priorities of alternatives in a manner that upsets original ratios, possibly leading to rank reversals. It is the overlooking of this intractable link between renormalization and criteria weights that accounts for the rank reversal problem. What one should take into consideration is the meaning of a criterion weight.

As shown by Schoner et. al. (1997) and Choo et. al. (1999), there is a necessary link between the normalization process and the weighting process. Initially in the hierarchical composition process, all local weights and priorities are normalized with respect to the totality of each set they are part of; the totality thus gets the unit value for many local ratio scales. In this respect, the local weights that sum to unity below each node can be visualized as a whole series of little hierarchies (node and sub-nodes) that have not yet been synthesised into the overall hierarchy (and an overall ratio scale). The unit weight of each sub-criterion pertains to the totality of the alternatives it is covering; the unit weight of each main criterion pertains to the totality of the sub-criteria it is covering.

It is subsequently the process of synthesis via hierarchical weighting that transforms the local weights into global weights that are in terms of a unit of the whole hierarchy. In effect, the former unit value that each node represented for the totality of directly covered items is re-scaled to be in terms of an overall hierarchical unit representing all alternatives and all criteria. Although this proportional transformation of local weights yields a new unit of measure, the global weights are still weighing the relative importance of the totalities they are referring to. The global weight of each sub-criterion pertains to the totality of the alternatives it is covering; similarly, the global weight of each main criterion pertains to the totality of the sub-criteria and alternatives it is covering.

## Maintaining a Benchmark Unit of Measure Upon Addition or Deletion of Alternatives

One way to maintain original ratios and therefore ranks with the addition of $A_3$ is to preserve the original unit of measure and allow the sum of all composite weights of the alternatives to add up to more than unity (e.g. $0.477+0.523+A_3>1$). In doing this, we would consider Figure 1 and its unit as a fixed, base hierarchy to which all other alternatives are evaluated. Rather than re-normalize local alternative priorities upon the addition of $A_3$, we could simply scale $A_3$ and any other new alternative so that it takes its relative value from existing alternatives that are deemed to have correctly established priorities. Those existing alternatives are benchmarks for establishing the priorities of other alternatives (Wedley et al., 1996). For example, the following situation derives $A_3$ without upsetting the ratio of $A_2/A_1$.

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{21}$ | $C_{22}$ |
|---|---|---|---|---|---|
| $A_1$: | 0.5 | 0.8 | 0.4 | 0.2 | 0.6 |
| $A_2$: | 0.5 | 0.2 | 0.6 | 0.8 | 0.4 |
| $A_3$: | 0.5 | 0.111 | 2.333 | 1.0 | 0.111 |

Notice that $A_1$ and $A_2$ have not been renormalized, that $A_3$ has been placed on the same relative scale and unit as $A_1$ and $A_2$, and that this leads to the sum of local weights being greater than 1. Composite priorities would now be:

$$A_1 = \{0.5*0.6+0.8*0.2+0.4*0.2\}*0.55 + \{0.2*0.5+0.6*0.5\}*0.45 = 0.477 \qquad (7)$$
$$A_2 = \{0.5*0.6+0.2*0.2+0.6*0.2\}*0.55 + \{0.8*0.5+0.4*0.5\}*0.45 = 0.523$$
$$A_3 = \{0.5*0.6+0.111*0.2+2.333*0.2\}*0.55 + \{1.0*0.5+0.111*0.5\}*0.45 = 0.684$$

The original composite units of $A_1$ and $A_2$ are unchanged, but the addition of $A_3$ has expanded the base hierarchy to a total of 1.684 units. This is primarily because $A_3$ is so strong on $C_{13}$. $A_3$ has, in fact, become the best choice, while $A_1$ and $A_2$ keep their original ratios and ranks. Had we renormalized to unit sum, then the incorrect priorities of (2) would have been reproduced.

## Adjusting Criteria Weights Upon Addition or Deletion of Alternatives

Another way to preserve rank is to rescale the new alternative into the sum-to-one convention (see Example 1), but recognize that the rescaling creates a new unit of measure for the hierarchy. The inclusion of $A_3$ changes the size of the hierarchical pie and we now must recognize that there is a link between the normalization process and criteria weights. Notice in the previous example, $A_3$ dramatically increases the supply of $C_{13}$. More $C_{13}$ and other criteria are brought to a hierarchy that gets its weights relative to items that are in it. With $A_3$ in $C_{13}$, for example, the amount of $C_{13}$ in the hierarchy has gone up by (1+2.333)/1 relative to its former self. The same applies to all other criteria. Accordingly, the global weights of criteria should be boosted by this relative amount and then renormalized to represent new priorities.

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{21}$ | $C_{22}$ |
|---|---|---|---|---|---|
| Adjustment | 1.5*.6*.55 | 1.111*.2*.55 | 3.333*.2*.55 | 2*.5*.45 | 1.111*.5*.45 |
| Adjusted value | 0.495 | 0.122 | 0.367 | 0.45 | 0.25 |
| Renormalized | 0.294 | 0.073 | 0.218 | 0.267 | 0.148 |

With these adjusted criteria weights, the composite priorities for the alternatives are:

$$A_1 = \{0.333*0.294+0.72*0.073+0.12*0.218\} + \{0.1*0.267+0.54*0.148\} = 0.283 \qquad (8)$$
$$A_2 = \{0.333*0.294+0.18*0.073+0.18*0.218\} + \{0.4*0.267+0.36*0.148\} = 0.311$$
$$A_3 = \{0.333*0.294+0.1 *0.073+0.7 *0.218\} + \{0.5*0.267+0.1 *0.148\} = 0.406$$

Again, $A_3$ is the best choice, $A_2$ is still better than $A_1$, and more importantly, the ratio $A_2/A_1$ is still 1.096. Both ratios and ranks are maintained if criteria weights are adjusted to account for the change in unit upon re-normalization.

## Fixed Relationship Between Criteria and Alternatives

A third way to preserve ratios and ranks upon addition or deletion of alternatives combines ideas from the previous two techniques. Rather than adjust criteria weights upon renormalization or fix on a benchmark unit of measure, a linking pin approach establishes a fixed relationship between criteria and alternatives (Schoner et al., 1993). Specific referent alternatives under each sub-criterion are given the value of unity and criteria weights are established for these specific alternatives. This relationship or link between criteria and alternatives is then kept constant. That way, the criteria weights that represent the value of referent alternatives can be distributed downwards to other alternatives via the ratio of the other alternatives to the referent alternative. In effect, referent alternatives become links to distribute the correct portion of each criterion to other alternatives. So long as this linking alternative remains in the hierarchy with its local priority of unity, it will continue to distribute hierarchical weights to other alternatives with no effect upon addition or deletion.

Although the best or ideal alternative of each criterion is usually selected as the link, this is not always necessary. For example, we can make $A_1$ the link under all $C_1$ and $A_2$ the link under all $C_2$.

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{21}$ | $C_{22}$ |
|---|---|---|---|---|---|
| $A_1$: | **1.0** | **1.0** | **1.0** | 0.25 | 1.5 |
| $A_2$: | 1.0 | 0.25 | 1.5 | **1.0** | **1.0** |

| | | | | | |
|---|---|---|---|---|---|
| $A_3$: | 1.0 | 0.139 | 5.833 | 1.25 | 0.278 |

Fixed global criteria priorities for these links are $c_i = (\hat{u}_{xi} / \Sigma_{x=1}^{2} u_{xi}) / \Sigma_{i=1}^{n} (\hat{u}_{xi} / \Sigma_{x=1}^{2} u_{xi})$ where $\hat{u}_{xi}$ is the selected reference link from $A_1$ or $A_2$. Relative global criteria weights established in reference to the links are derived as follows using the original priorities of $A_1$ and $A_2$ respectively (ref. Figure 1):

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{21}$ | $C_{22}$ |
|---|---|---|---|---|---|
| Adjustment | (0.5/1)*0.33 | (0.8/1)*0.11 | (0.4/1)*0.11 | (0.8/1)*0.225 | (0.4/1)*0.225 |
| Adjusted value | 0.165 | 0.088 | 0.044 | 0.18 | 0.09 |
| Renormalized | 0.291 | 0.155 | 0.078 | 0.317 | 0.159 |

With this fixed relationship, the composite priorities for $A_1$ and $A_2$ are 0.841 and 0.922 respectively. $A_2$ is 0.922/0.841 = 1.096 times better than $A_1$, as it should be. When $A_3$ is added, it assumes its priority ($A_3$=1.206) via the fixed relationship between criteria and referent alternatives. So long as the referent links do not change, additions or deletions have no effect on other alternatives. If reference alternatives are changed or removed, then it would be necessary to establish new criteria weights for the fixed relationship between alternatives and criteria. To avoid this, we recommend referent alternatives be kept fixed.

### Avoiding Reversals on Removal of Non-Discriminating Criteria

The cause of reversals when removing non-discriminating criteria from a multilevel hierarchy is similar to reversals that occur on addition or deletion of alternatives. Removal of a wash criterion is the same as removal of a portion of the hierarchical unit. If renormalization takes place thereafter, the former unit of measurement is changed and ranks can change.

### Maintaining a Benchmark Unit Upon Removal of a Criterion

Had we not re-normalized after removing the non-discriminating criterion in (3), then the original unit of measure would be maintained and the composite results would be:

$$A_1 = \{0.8*0.2+0.4*0.2\}*0.55 + \{0.2*0.5+0.6*0.5\}*0.45 = 0.312 \qquad (10)$$
$$A_2 = \{0.2*0.2+0.6*0.2\}*0.55 + \{0.8*0.5+0.4*0.5\}*0.45 = 0.358$$

We notice that these results with ranks maintained are identical to those obtained in (5) by similarly not re-normalizing global weights ($A_2/A_1$=0.358/0.312=1.147). In (10), we kept the original normalizations as benchmarks. The removal of $C_{11}$ heightened the importance of $A_2$, as we would expect, and the same overall unit maintained ranks, as we would also expect.

Finan and Hurley's unexpected reversal in (3) did not account for the necessary link between the normalization process and the weighting process (Choo et al, 1999). Instead, they used unit sum re-normalization of the local weights without realizing that the unit of that sum has changed with the removal of the sub-criterion. With $C_{11}$ removed, local unit change upon re-normalization distorts the ratio of the global weights of $C_1$'s criteria set with respect to those of the $C_2$ set, which keep their original weights. This distortion offsets the expected accentuation of the best alternative ($A_2$) and causes a reversal.

## Adjusting Criteria Weights Upon Removal of a Criterion.

When deriving local criteria weights, it is important to consider the totality of the items below as a reference. Just as the criteria weights should be adjusted upon the re-normalization of a new set of alternatives, so too should they be adjusted upon re-normalization with a smaller set of criteria. A change in the set of sub-criteria is the case in Finan & Hurley's example reproduced in (3). Leaving out $C_{11}$ changes $C_1$'s set of sub-criteria; the new set now is 0.2+0.2=0.4 of its former self. The main criterion $C_1$ now represents a different totality of sub-criteria below. Accordingly, the weight of $C_1$ should be adjusted to 0.55*0.4=0.22 in order to safely re-normalize the remaining local weights of its sub-set to the unit sum.

The adjusted criterion weight for $C_1$ (0.22) remains commensurate to $C_2$ (0.45), although they do not sum to one. Synthesis using the new, normalized local weights of $C_{12}$ and $C_{13}$ (both 0.5) and the adjusted weight for $C_1$ produces:

$$A_1 = \{0.8*0.5+0.4*0.5\}*0.22 + \{0.2*0.5+0.6*0.5\}*0.45 = 0.312 \quad (11)$$
$$A_2 = \{0.2*0.5+0.6*0.5\}*0.22 + \{0.8*0.5+0.4*0.5\}*0.45 = 0.358$$

We could have $C_1$=0.22 and $C_2$=0.45 re-normalized to sum to one, but this would not have changed the preference ratio between the two alternatives. $A_2$ is 1.147 times preferred to $A_1$ which is the result we would want it to be.

The new values of the global weights can be computed using $C_1$'s adjusted weight and the new, normalized local weights of $C_{12}$ and $C_{13}$ (both 0.5):

$$C_{12} = C_{13} = 0.22*0.5 = 0.11$$
$$C_{21} = C_{22} = 0.45*0.5 = 0.225$$

The global weights of $C_{12}$, $C_{13}$, $C_{21}$ and $C_{22}$ have not changed and neither have the local priorities of the alternatives. Synthesis again yields composite priorities of 0.312 for $A_1$ and 0.358 for $A_2$ with $A_2$ 1.147 times more preferred than $A_1$, therefore not showing rank reversal.

Had $C_1$=0.22 and $C_2$=0.45 been re-normalized to sum to one, then the result $C_1$=0.328 and $C_2$=0.672 would still be commensurate, but with a different unit of measure. Synthesis using the new, normalized local weights of $C_{12}$ and $C_{13}$ (both 0.5) and the re-normalized weight for $C_1$ and $C_2$ (after adjustment of $C_1$'s weight) produces:

$$A_1 = \{0.8*0.5+0.4*0.5\}*0.328 + \{0.2*0.5+0.6*0.5\}*0.672 = 0.4657 \quad (12)$$
$$A_2 = \{0.2*0.5+0.6*0.5\}*0.328 + \{0.8*0.5+0.4*0.5\}*0.672 = 0.5343$$

With global weights, computed from adjusted, fully re-normalized local weights, the results would be identical to those in (6) where the correct ratio of 1.147 is achieved.

## Observations and Conclusion

Of the two types of reversals, we question whether a non-discriminating criterion should be removed. Our techniques demonstrate that removal of a non-discriminating criterion accentuates the superiority of the best alternative while maintaining its top rank.

Since its rank remains unchanged, there is no advantage in removing the criterion except to make the best alternative look better. Such optics should be avoided, because if the wash criterion was relevant in the first place, then it should be maintained as relevant to the final solution. Besides, removing a non-discriminating criterion can be dangerous. If the relative priorities of all alternatives are used to allocate resources, the changed ratios upon removal of a wash criterion can result in distorted allocations. For the purpose of scientific integrity, our preference is to diligently define and structure the problem at the outset and then use all relevant criteria to make the decision, including those that turn out to be non-discriminate.

We note that it is rare for weights to be adjusted upon a change of a set of elements in an AHP hierarchy. This is not surprising as AHP's axiom 3, the "independence axiom" (Saaty, 2000), does not formally require criteria weights to be derived in relation to lower-level elements and their normalization. The independence axiom states that elements on a specific hierarchy level are dependent on their parent-elements on the next higher hierarchy level, but independent of their child-elements (criteria or alternatives) on the next lower level. We question whether this ever occurs in reality.

We believe that it is better to assume dependence and deal with it properly than to assume independence and face phenomena like rank reversal that are difficult to justify or explain. With the Analytic Network Process, this is essentially what happens (Saaty, 1996). In this paper, we have shown that assuming dependence and acting accordingly prevents undue rank reversal when an alternative is added or deleted or when a non-discriminating criterion is removed. The latter situation is particularly striking and has the flavour of the rank reversal problem of Belton & Gear (1983) where a copy of an existing alternative was added to the choice set. In both cases, the totality of a set of hierarchy elements was changed, thereby changing the unit of that totality when re-normalising to the unit sum. Incidentally, that totality will change regardless of the non-discriminating nature of a criterion being removed or the identical nature of an alternative being added. As soon as one or more of the sets of hierarchy elements are changed by adding or removing elements, the appropriate local weights must be re-considered and probably adjusted to maintain commensurateness and thus prevent ranks from reversing. In the case of benchmark or linking pin synthesis where the unit of measure is fixed, such re-adjustment is unnecessary (unless the deletion is the benchmark or linking pin).

It should be pointed out that, so far, we have used the weaker requirement of rank preservation rather than ratio preservation that could be applied to AHP. As shown above, the deletion of a criterion can cause composite ratios to change, although proper weight adjustment preserves rank. However, if an alternative is added or deleted, the weight adjustment we have suggested would not have resulted in a change in ratios. The reason for this differential effect is that the global priorities of the alternatives on each criterion are in commensurate units. If the alternatives are discrete and independent of one another, then summation across all criteria does not upset the ratio between existing alternatives. If, however, we sum across different sub-sets of criteria, the results will be different ratios. In effect, each addition or deletion of criteria presents a new evaluation problem whereas addition or deletion of an alternative just changes the choice set in the same evaluation problem. In both cases, however, the appropriate techniques or adjustments are available to maintain the integrity of the ranks.

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