$\mathcal{H}_2$-optimal Control of an Adaptive Optics System: Part I, Data-driven modeling of the wavefront disturbance

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ABSTRACT

Even though the wavefront distortion introduced by atmospheric turbulence is a dynamic process, its temporal evolution is usually neglected in the adaptive optics (AO) control design. Most AO control systems consider only the spatial correlation in a separate wavefront reconstruction step. By accounting for the temporal evolution of the wavefront it should be possible to further reduce the residual phase error and enable the use of fainter guide stars. Designing a controller that takes full advantage of the spatio-temporal correlation in the wavefront requires a detailed model of the wavefront distortion. In this paper we present a dedicated subspace identification algorithm that is able to provide the required prior knowledge. On the basis of open-loop wavefront slope data it estimates a multi-variable state-space model of the wavefront disturbance. The model provides a full description of the spatio-temporal statistics in a form that is suitable for control. The algorithm is demonstrated on open-loop wavefront data.

Keywords: Adaptive optics, optimal control, spatial-temporal correlation, data-driven disturbance modeling

1. INTRODUCTION

The control strategy used in the current generation of AO system is generally not able to take full advantage of the spatio-temporal correlation in the wavefront disturbance. In standard AO control design[1,2] it is even not uncommon to entirely neglect the temporal correlation in the wavefront. Based on physical insights, the standard control law consists of a cascade of a static matrix multiplication and a series of parallel single-input single-output (SISO) feedback loops. Given at one time instance a new wavefront sensor (WFS) measurement, the problem addressed in standard AO control is that of finding (estimating) the deformable mirror (DM) actuator input that would provide the best fit to the measured wavefront. Both maximum likelihood and maximum a posteriori techniques are used to improve the accuracy of this estimate, however the modified wavefront statistics due to closed loop operation is usually neglected. The parallel feedback loops, on the other hand, act as a servo controller and are responsible for stability and closed-loop performance. In this perspective it is important to note that the WFS measures the residual and not the open-loop wavefront. The actuator inputs generated by the static matrix multiplication provide only an estimate of the correction increment that has to be applied to the current actuator input and are not able to compensate the wavefront itself. To deal with this shortcoming the servo controller has to posses integrating action. The individual feedback loops usually have a predefined structure which has its roots in classical control theory. Common servo controller structures include the leaky integrator, the PI controller and the Smith predictor. The control parameters, which are responsible for the trade off between disturbance rejection, noise propagation and closed-loop stability, are usually determined on the basis of heuristic tuning rules. No specific knowledge of the temporal aspects of the wavefront disturbance is used.

From the above discussion it is to be expected that a lot can be gained by using a rigorous control strategy that is able to account for both the dynamics of the wavefront and AO system components and the modified statistics due to the closed loop behavior. The performance of an AO control system is usually limited by the presence of measurement noise and the time delay between measurement and correction. By including the temporal aspect it is possible to anticipate on future distortions and reduce the effect of the time delay. The

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close resemblance between wavefront prediction and optimal control has been pointed out in our previous work.[3] Also the sensitivity to measurement noise may be reduced by including the spatio-temporal correlation. Photons from different time instants and different WFS channels may all contribute to improve the wavefront estimate at a certain position in the aperture plane. When the frozen layer hypothesis is satisfied for example, neighboring WFS channels on the lee side provide direct information of the future development of the turbulence. The idea to benefit from the temporal correlation has already been explored in literature. The modal control optimization approach[4, 5] is an important example in which the wavefront is decomposed in a set of modes of which the servo bandwidth are individually tuned. A disadvantage of this approach is that spatial and temporal dynamics are assumed to be decoupled and the performance depends on the chosen modal decomposition.

In this work the AO control problem is analyzed from a control perspective. The problem is conceived as a multi-variable discrete-time disturbance rejection problem. With the objective of minimizing the residual variance, the problem will be formulated in a $\mathcal{H}_2$-optimal control framework. This approach bears a close resemblance with linear quadratic Gaussian (LQG) control and the use of optimal Kalman filters,[6–9] but it provides a more elegant way to deal with the fact that the WFS measures slopes while it is the residual phase that we try to minimize. Except for the discussion by Gavel and Wiberg,[9] which relies on the frozen frozen layer hypothesis, each of the LQG approaches uses a rather restrictive disturbance model which still assumes that the spatial and temporal dynamics can be decoupled. In this paper we present a dedicated subspace identification algorithm that is able to identify a control-relevant disturbance model for AO systems of moderate size. The disturbance model will be estimated on the basis of open-loop WFS measurements and provides a full description of the spatio-temporal behavior of the wavefront. Data-driven modeling has the advantage that it yields a good match with the prevalent turbulence conditions and it results in a disturbance model independent from assumptions like the frozen flow hypothesis. With the identified disturbance model it is possible to compute a controller that is able to exploit the spatio-temporal correlation in the wavefront. The problem of solving the $\mathcal{H}_2$ optimal control problem will be considered in the companion paper.[10]

The remainder of this paper is organized as follows. In Section 2 we provide an accurate description of the stochastic identification problem considered in this paper. Apart from explaining the problem this section introduces a projection that will be used to account for the fact that only a part of the wavefront can be reconstructed from the WFS measurements. In Section 3 we present a dedicated subspace identification algorithm that is able to provide a control-relevant disturbance model. The proposed identification method has been demonstrated on the basis of both open-loop WFS obtained from a breadboard setup and real-live data from the JOSE seeing-monitor at the William Herschel telescope. The results of these validation experiments are reported in Section 4. The paper concludes with a short discussion in Section 5.

2. PROBLEM FORMULATION

Computing the $\mathcal{H}_2$-optimal controller designed in companion paper[10] requires accurate knowledge of second order statistics of the uncompensated wavefront and the corresponding WFS signal. The stochastic identification problem considered in this paper is concerned with the problem of determining a control relevant disturbance model that is able to provide this information. It will be assumed that the phase distortion profile can be represented by a finite dimensional discrete-time vector signal ϕ_k, where the subscript k denotes the time index. Whether this representation is obtained by a zonal or modal representation is irrelevant as long as the variance of the vector signal $\phi_k \in \mathbb{R}^{m_p}$ provides a good approximation of the phase variance over the aperture.

An important issue in the AO control problem is that the wavefront distortion ϕ_k can not be measured directly. Only the WFS slope measurements $s_k \in \mathbb{R}^{m_s}$ are available for identification and control. It is generally not possible to reconstruct the entire wavefront from this signal. In order to arrive at a well-posed identification and control problem the unobservable part of the wavefront should be disregarded. We will assume that the optical transformation from phase distortion profile ϕ_k to wavefront slope measurements s_k is described by:

$$s_k = G\phi_k + \eta_k, \tag{1}$$

where $G \in \mathbb{R}^{m_s \times m_p}$ is the geometry or phase-to-slope influence matrix and η_k is a zero-mean white noise term which represents the measuring noise. Any dynamics introduced by the wavefront sensor is absorbed in the stochastic disturbance model. Since the disturbance model should describe the second order statistics of the WFS signal s_k, this is not very restrictive. The fact that only a part of the wavefront can be reconstructed from the slope measurements manifest itself in a rank deficiency of the matrix G. The unobservable part of the wavefront ϕ_k can be removed by considering the singular value decomposition (SVD) of matrix G:

$$G = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \tag{2}$$

where $U \in \mathbb{R}^{m_s \times m_s}$ and $V \in \mathbb{R}^{m_p \times m_p}$ are orthogonal matrices and the partitioning of the diagonal matrix $\Sigma \in \mathbb{R}^{m_s \times m_p}$ is such that the matrix $\Sigma_1 \in \mathbb{R}^{m_r \times m_r}$ contains all nonzero singular values. By substituting the SVD in equation (1) and pre-multiplying both sides with the matrix U_1^T, we obtain the following reduced WFS model

$$r_k = \Sigma_1 \psi_k + U_1^T \eta_k, \tag{3}$$

where $r_k \triangleq U_1^T s_k \in \mathbb{R}^{m_r}$ and $\psi_k \triangleq V_1^T \phi_k \in \mathbb{R}^{m_r}$. The newly defined signal ψ_k provides a reduced representation of the observable part of the wavefront ϕ_k with respect to the basis V_1. This can be seen by noting that through the orthogonality of the matrix V the wavefront can be decomposed as $\phi_k = V_1 V_1^T \phi_k + V_2 V_2^T \phi_k$. Whereas the first term has a direct influence on the measured output signal s_k, the second term cannot be observed as it lies in the null-space of the matrix G. Another consequence of the orthogonality of the matrix V is that the observable part of the wavefront $V_1 V_1^T \phi_k = V_1 \psi_k$ and the signal ψ_k have the same Euclidean norm. This implies that we can replace the role of the open-loop wavefront distortion ϕ_k by the reduced representation ψ_k when the control objective is expressed in terms of the Euclidean norm. Similarly, the signal r_k can be used as a reduced representation of the WFS signal s_k, where it is interesting to note that projection U_1^T removes the part of the measuring noise that is not in the range space of G and cannot be caused by the wavefront. Furthermore, since the matrix G is generally tall, the proposed projection results in a significant reduction of the signal that has to be modeled.

Since the statistical properties of the wavefront change on a time scale that is long compared to the time scale of the fluctuations themselves, it is reasonable to conceive the reduced WFS signal r_k as a wide sense stationary (WSS) process. Looking at the wavefront disturbance from this perspective shows that it should be possible to identify a disturbance model from open-loop WFS data and use this model to compute a controller that is optimal over a certain time horizon. To guarantee a good performance over longer time periods the disturbance model should be updated on a regular basis. In this way it is still possible to account for the changing wavefront statistics that occur in the course of the night. By considering the reduced WFS signal as WSS, the signal can be modeled as the output of a stable linear time invariant (LTI) system with a zero-mean white noise input $v_k \in \mathbb{R}^{m_r}$. Each element of the vector signal is represented by a separate output channel. This in combination with the reduced WFS model (3) motivates us to introduce the following model structure for the disturbance model

$$S : \left\{ \begin{array}{rcrcl} x_{k+1} &=& A\,x_k &+& K v_k \\ r_k &=& \Sigma_1 C\,x_k &+& v_k \\ \psi_k &=& C\,x_k &+& e_k \end{array} \right. , \qquad \mathcal{E}\left(\begin{bmatrix} v_k \\ e_k \\ I \end{bmatrix} \begin{bmatrix} v_l^T & e_l^T \end{bmatrix} \right) = \begin{bmatrix} R_{ve} \\ 0 \end{bmatrix} \delta_{kl}, \tag{4}$$

where $e_k (\triangleq \Sigma^{-1}(v_k - U_1^T \eta_k))$ is again zero-mean white noise signal and the measurement noise η_k is included in the state-space representation of the WFS output r_k. The proposed model structure is in innovation form with respect to this output. This implies that the mapping from the white noise input v_k to the WFS signal has a stable inverse, which is a very useful property for controller design. In control and system theory, a state-space representation S that provides a stably invertible description of the stochastic process is commonly referred to as the minimum-phase spectral factor of the process.

Since Σ_1 is known from the SVD (2), the calculation of the $\mathcal{H}_2$-optimal controller, developed in,[10] only requires the estimation of the state space matrices $A \in \mathbb{R}^{n \times n}$, $\Sigma_1 C \in \mathbb{R}^{m_r \times n}$ and $K \in \mathbb{R}^{m_r \times m_r}$ from open-loop observations of r_k.

3. DATA-DRIVEN IDENTIFICATION OF A DISTURBANCE MODEL

In this section we present a dedicated subspace identification algorithm that is able to deal with the stochastic identification problem described before. Based on open-loop data, the algorithm provides a full statio-temporal disturbance model for AO systems of moderate size. Before considering the algorithm, we will briefly examine the motivation to use a subspace based identification approach.

3.1. Motivation for a subspace approach to disturbance modeling

As explained in Section 2, each element of the reduced WFS signal r_k is represented as a separate output of a large LTI system with zero-mean white noise input. The model structure S, obtained by incorporating the reduced WFS model, is able to respect the mutual correlation that exists between each of the reduced WFS channels r_k and the observable part of the phase ϕ_k. It provides a complete characterization of the second order statistics of and between each the channels. A possible complication of this extensive description is that even for relatively small AO systems this leads to a rather large-scale identification problem. For this reason it is extremely important the have an efficient identification algorithm.

Based on numerically robust matrix operations, subspace algorithms bypass the need for model parameterization and non-linear optimization. This is an important advantage over the more traditional maximum likelihood and prediction error methods,[11] which rely on the optimization of structural parameters over a suitably chosen cost-function. Except for some rather restrictive parameterizations, these methods give rise to a non-linear optimization problem. Apart from the danger of ending up in a local minimum, the computational load required to search the parameter space increases rapidly with the number of variables. This is especially a problem in the multivariate case where the mapping from each input to each output is usually parametrized independently. Subspace identification, on the other hand, is able to treat the multivariate case within the same framework, without any fundamental difficulties.

The subspace algorithm presented in this paper is basically an output only version of the SSARX algorithm proposed by Jansson[12] and is closely related to the canonical correlation analysis method.[13] In analogy with nomenclature introduced by Jansson, the algorithm will be referred to as SSAR. An important advantage of the proposed algorithm is that it does not rely on the solution of a Riccati equation to perform a spectral factorization. Most subspace algorithms for stochastic identification use a two-stage procedure which consists of estimating a rational covariance model followed by a spectral factorization step. As demonstrated by Lindquist and Picci,[14] this procedure might easily fail since there is no guarantee that the estimated covariance model has a valid spectral factor. The subspace algorithm used in this paper avoids this problem. By using the coefficients of a high order autoregressive (AR) model, more structure is added to the data model used by the subspace algorithm. This opens the possibility to estimate the state-space matrices $A, \Sigma_1 C$ and K directly, without the intermediate step of estimating a rational covariance model. Another advantage of the presented subspace algorithm is that it can be extended to the case that the data is collected in closed-loop. This is especially important since this may be used to update the disturbance model without interrupting the observations.

The computational demands of the proposed subspace algorithm have been reduced by using an efficient implementation. A single RQ factorization of the stacked block-Hankel matrix is used both to compute the required AR coefficients and for data compression. In computing this RQ factorization it is possible to exploit the displacement structure of the block-Hankel matrix. This is achieved by using the algorithm by Mastronardi et al.[15] The efficient implementation of the algorithm is beyond the scope of this paper.

3.2. The SSAR stochastic subspace identification method

The subspace identification algorithm used to identify the disturbance model from the open-loop measurement data is based on an alternative model representation of the open-loop WFS signal $r_k \in \mathbb{R}^{m_r}$. Consider the model structure introduced in equation (4). By using the output equation for r_k it is possible to eliminate the noise term v_k from the state update equation. This gives rise to the following equivalent representation:

$$\begin{cases} x_{k+1} &= \widetilde{A}x_k + Kr_k \\ r_k &= \widetilde{C}x_k + v_k, \end{cases} \tag{5}$$

where $\widetilde{A} \triangleq A - K\widetilde{C}$ and $\widetilde{C} \triangleq \Sigma_1 C$. The state space representation (4) is required to be stable and strictly minimum phase with respect to the output signal r_k. The latter condition implies that $|\lambda_i(A - K\Sigma_1 C)| = |\lambda_i(\widetilde{A})| < 1,\ \forall i \in \{1, \ldots, n\}$, where $\lambda_i(\cdot)$ denotes the i^{th} eigenvalue. The minimum phase condition on (4) is therefore equivalent to demanding that the alternative model representation (5) is stable. Furthermore it is assumed that the matrix pairs $\{A, K\}$ and $\{A, C\}$ are controllable and observable,[16] respectively. By recursive application of the state update equation in (5), it is easy to show that the state of the system at a certain time instant k can be expressed as:

$$x_k = \widetilde{A}^p x_{k-p} + \sum_{j=0}^{p-1} \widetilde{A}^j K r_{k-j-1}. \tag{6}$$

For stable $\widetilde{A}$ and sufficiently large $p \in \mathbb{N}$, the first term in equation (6) can be neglected and the state x_k can be approximated by a linear combination of past open-loop WFS data r_k. In matrix notation, this state estimate can be expressed as:

$$\hat{x}_k = \mathcal{K} \begin{bmatrix} r_{k-p} \\ r_{k-p+1} \\ \vdots \\ r_{k-1} \end{bmatrix}, \qquad \text{where} \quad \mathcal{K} \triangleq \begin{bmatrix} \widetilde{A}^{p-1} K & \dots & \widetilde{A} K & K \end{bmatrix}. \tag{7}$$

Given the state at a time instant k, equation (5) can be employed to derive the following relation between the data batch of present and future innovations $\{v_j\}_{j=k}^{k+q+1}$ on the one hand and the data batch of present and future open-loop WFS measurements $\{r_j\}_{j=k}^{j=k+q-1}$ on the other hand:

$$\left[\begin{array}{c} r_k \\ \hline r_{k+1} \\ r_{k+2} \\ \vdots \\ r_{k+q+1} \end{array}\right] = \underbrace{\left[\begin{array}{c} \widetilde{C} \\ \hline \widetilde{C}\widetilde{A} \\ \widetilde{C}\widetilde{A}^2 \\ \vdots \\ \widetilde{C}\widetilde{A}^{q-1} \end{array}\right]}_{\Gamma} x_k + \underbrace{\left[\begin{array}{c|cccc} 0 & 0 & \dots & 0 & 0 \\ \hline \widetilde{C}K & 0 & \dots & 0 & 0 \\ \widetilde{C}\widetilde{A}K & \widetilde{C}K & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ \widetilde{C}\widetilde{A}^{q-2}K & \dots & \dots & \widetilde{C}K & 0 \end{array}\right]}_{\Psi} \left[\begin{array}{c} r_k \\ \hline r_{k+1} \\ r_{k+2} \\ \vdots \\ r_{k+q+1} \end{array}\right] + \left[\begin{array}{c} v_k \\ \hline v_{k+1} \\ v_{k+2} \\ \vdots \\ v_{k+q+1} \end{array}\right], \tag{8}$$

where $q \in \mathbb{N}$ is the number of block-Hankel rows which should be large enough to avoid singularity problems in estimating the system matrices $\tilde{A}$ and $\tilde{C}$. More specifically, the data equation (8) should contain sufficient block-rows to guarantee that the matrix Γ has full column rank.[12] With the pair $\{A, C\}$ being observable, a sufficient condition to satisfy this requirement is to demand that the parameter q is larger than the system order n. For stochastic systems with large numbers of outputs this condition may be overly conservative. As we will see in the numerical validation experiments in Section 4, it may be useful to choose the parameter q much smaller than the system order that is eventually used to describe the process. A same reasoning holds for the parameter p in (7). This parameter should be chosen sufficiently large to guarantee that the matrix $\mathcal{K}$ has full row rank.

The partitioning of the matrices and vectors in equation (8) is such that there a clear distinction between present and future data samples. By substituting the state estimate (7), we obtain an expression that relates vectors derived from past and future open-loop WFS and innovation data to the matrices $\mathcal{K}$, Γ and Ψ derived from the system matrices $(\widetilde{A}, \widetilde{C}, K)$. Since the underlying system is assumed to be time-invariant, the above equations hold also for time-shifted versions of the state, open-loop WFS and innovation vectors. It is therefore possible to relate the unknown matrices $\Gamma\mathcal{K}, \Psi$ to matrices composed of time-shifted columns of open-loop WFS and innovation data. To represent this matrix relation in a compact way, let us introduce the block Hankel matrix

$$H_{i,q,N} \triangleq \begin{bmatrix} r_i & r_{i+1} & \dots & r_{i+N-1} \\ r_{i+1} & r_{i+2} & \dots & r_{i+N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{i+q-1} & r_{i+q} & \dots & r_{i+N+q-2} \end{bmatrix}, \tag{9}$$

where the first subscript refers to the time-index of the top left entry, the second refers to the number of block-rows and the third represents the number of columns. The number of columns depends on the available amount of data and is generally such that $q \ll N$. Using the same notational convention, the block Hankel matrix constructed from innovation data v_k will be denoted by $V_{i,q,N}$. In this notation, the data equation obtained by combining time-shifted versions of equation (7) and (8) can be expressed as:

$$(I - \Psi)H_{k,q,N} \approx \Gamma\mathcal{K}H_{k-p,p,N} + V_{k,q,N} \tag{10}$$

Equation (10) forms the basis of the proposed subspace identification algorithm, which starts with estimating the matrix $\widetilde{C}\mathcal{K}$. To estimate the matrix $\widetilde{C}\mathcal{K}$ from the measurement data r_k, it is important to recognize that first block-row of the matrix Ψ is zero. Since the first block-row of the matrix Γ is equal to $\widetilde{C}$, it follows from equation (10) that

$$H_{k,1,N} \approx \widetilde{C}\mathcal{K}H_{k-p,p,N} + V_{k,1,N}. \tag{11}$$

This equation expresses the present open-loop WFS values as a linear combination of past data and the present innovation and is therefore nothing else than a high-order autoregressive (AR) model of the open-loop WFS signal. Since the innovation signal v_k is a zero mean white noise sequence, its current value is uncorrelated with past open-loop WFS data. As a result, an unbiased estimate of the matrix $\widetilde{C}\mathcal{K}$ can be obtained as the argument that solves the least squares problem

$$\widehat{\widetilde{C}\mathcal{K}} = \arg\min_{\widetilde{C}\mathcal{K}} \|H_{1,k,N} - (\widetilde{C}\mathcal{K})H_{k-p,p,N}\|_2^2. \tag{12}$$

From the above interpretation of equation (11), it is clear that the matrix $\widetilde{C}\mathcal{K}$ provides a direct estimate of the AR coefficients $\widetilde{C}\widetilde{A}^j K,\ j \in \{0, 1, \ldots, p-1\}$. This observation is used in the next step of the subspace identification algorithm to form an estimate of the matrix Ψ. Like the matrix $C\mathcal{K}$, Φ is fully determined by the first few AR coefficients of the stochastic process. When the parameter p is chosen such that $p \geq q$, an estimate of the matrix Ψ can be simply obtained by extracting the AR coefficients for the matrix $\widetilde{C}\mathcal{K}$ and arranging them in the proper form. This estimate becomes consistent when the parameters p is allowed to increase. By substituting $\widehat{\Psi}$ in equation (10), more structure is added to the data equation, which enables us to make a clear distinction between the effect of future and past open-loop WFS data r_k. This results again a linear matrix equation, perturbed by a zero mean white noise, from which the matrix $\Gamma\mathcal{K}$ can be obtained by solving the least squares problem

$$\widehat{\Gamma\mathcal{K}} = \arg\min_{\Gamma\mathcal{K}} \|(I - \Psi)H_{k,p,N} - (\Gamma\mathcal{K})H_{k-p,p,N}\|_2^2. \tag{13}$$

When the parameters p and q (with $p \geq q$) are sufficiently large, the matrices $\mathcal{K}$ and Γ are both full rank. This implies that can be factorized to determine the column space and row space of the matrices Γ and $\mathcal{K}$, respectively. Let the SVD of the matrix $\widehat{\Gamma\mathcal{K}}$ be given by

$$\widehat{\Gamma\mathcal{K}} = U_n\Sigma_n V_n^T, \tag{14}$$

where the matrix $\Sigma_n \in \mathbb{R}^{n\times n}$ contains the dominant singular values and U_n and V_n^T denote the matrices composed of the corresponding left and right singular vectors. Since the rank of the matrix $\Gamma\mathcal{K}$ is known to be n, the number of singular values that differ significantly from zero can be used as an estimate of the system order n. Furthermore, the left and right singular matrices U_n and V_n of the SVD can be used to estimate the matrices Γ and $\mathcal{K}$ up to an unknown non-singular similarity transformation $T \in \mathbb{R}^{n\times n}$, i.e.

$$U_n\Sigma^{1/2} = \widehat{\Gamma}T = \begin{bmatrix} C_T \\ C_T A_T \\ \vdots \\ C_T A_T^{q-1} \end{bmatrix}, \qquad \Sigma^{1/2}V_n^T = T^{-1}\widehat{\mathcal{K}} = \begin{bmatrix} \widetilde{A}_T^{p-1}K_T & \cdots & \widetilde{A}_T K_T & K_T \end{bmatrix}, \tag{15}$$

where $(A_T, C_T, K_T) = (T^{-1}AT, CT, T^{-1}K)$. The non-uniqueness, expressed by the presence of the similarity transformation T, is caused by the freedom in choosing the basis for the column and row space of $\widehat{\Gamma\mathcal{K}}$. All possible bases however, result in a similarity-equivalent triple (A_T, C_T, K_T). The particular choice of T has no influence on the mapping from the innovation signal v_k to the open-loop WFS signal r_k define by (5). An estimate of the matrix C can now be obtained by extracting the first m_r rows from the matrix $U_n\Sigma^{1/2}$. Furthermore, the matrix

K is estimated by selecting the last m_r columns of the matrix $\Sigma^{1/2}V_n^T$. The matrix A is finally estimated by solving the overdetermined equation

$$(\overline{U_n}\Sigma^{1/2})\widehat{\widetilde{A}} = \underline{U_n}\Sigma^{1/2}, \tag{16}$$

where $\overline{U_n}$ and $\underline{U_n}$ denote the matrices obtained by deleting, respectively, the m_r first and last rows of U_n. The estimated triple $(\widetilde{A}, \widetilde{C}, K)$ in combination with the singular value matrix $\Sigma_1 \in \mathbb{R}^{m_r \times m_r}$ provides a complete specification of the atmospheric disturbance model (4) needed to compute the closed-loop optimal controller.[10] Using the definition of the matrices $\widetilde{A}$ and $\widetilde{C}$, it is possible to explicitly compute A and C. This last step may even not be necessary in some cases. As will be demonstrated in Section 3 of the companion paper[10] on controller design, the $\mathcal{H}_2$-optimal controller of an AO system that is quasi static in the sense that the WFS and DM are static expect for a unit delay, can be conveniently expressed in terms of the identified matrices $\widetilde{A}, \widetilde{C}$ and K. Also for the numerical validation experiments described in Section 4, it is not necessary to explicitly compute the matrices A and C.

4. NUMERICAL VALIDATION ON OPEN-LOOP DATA

The wavefront disturbance modeling approach outlined in Section 3.2 has been validated in two realistic measurement scenarios. The first scenario comprises of a breadboard setup developed by the Dutch company TNO Science and Industry and the second scenario corresponds to real-life measurements taken from the JOSE seeing monitor at the William Herschel telescope. The latter data set was also used in our previous paper.[3] The main difference with the current approach is that disturbance modeling is made fully compatible with the $\mathcal{H}_2$-optimal controller design and that all channels of the reduced WFS signal r_k and the observable part of the wavefront ψ_k are modeled simultaneously. Part of the wavefront reconstruction is therefore already performed in the identification of an atmospheric disturbance model. This opens the possibility to formulate the $\mathcal{H}_2$-optimal control problem without first considering a wavefront reconstruction step.[10] Both the spatial and temporal behavior of the uncorrected wavefront distortion are modeled on the basis of the available WFS measurements only.

The performance of the identified disturbance model is evaluated by considering the one-step ahead prediction problem. For the outlined disturbance modeling approach in Section 3.2, the one-step ahead prediction of the ψ_k is readily derived from the identified innovation model (5) and given by[11]:,

$$\mathcal{F}^{\mathrm{KP}} : \left\{ \begin{array}{lcl} x_{k+1} & = & \tilde{A}x_k \quad + \quad Kr_k \\ \hat{r}_k & = & \tilde{C}x_k \\ \hat{\psi}_k & = & Cx_k \end{array} \right. \tag{17}$$

Using equation (17) the wavefront prediction error can be computed as $\varepsilon_k = \hat{\psi}_k - \psi_k = \mathcal{F}^{\mathrm{KP}} r_k - \Sigma^{-1} r_k$. The prediction error is compared with the random walk approach. In the random walk approach the best estimate of the wavefront at next sample instant is given by the wavefront reconstructed from the current WFS measurement. This is equivalent to saying that $\hat{r}_{k+1} = r_k$, which in analogy with (17), leads to the following state space prediction model:

$$\mathcal{F}^{\mathrm{RW}} : \left\{ \begin{array}{lcl} x_{k+1} & = & r_k \\ \hat{r}_k & = & x_k \\ \hat{\psi}_k & = & \Sigma_1^{-1} x_k \end{array} \right. , \tag{18}$$

The random walk approach has a close resemblance with the common AO law, where the time delay between measurement and correction is neglected. Remark that the state dimension of the latter predictor equals the dimension of the reduced WFS signal.

The criterion used to evaluate the performance of the identified wavefront disturbance model is explicitly defined in equation (19). It represents the average of the mean square error of the one-step ahead prediction normalized to the space and time averaged mean square value of the observable part of the wavefront ϕ_k. Let the j-th component of the observable part of the wavefront signal at time instant k be denoted by $\psi_k(j)$, and let the total number of time measurements be denoted by N, then the considered performance criteria are given by:

$$J_1 = \frac{\sum_{j=1}^{n_r}\sum_{k=1}^{N-1}(\hat{\psi}_{k+1}(j) - \psi_{k+1}(j))^2}{\sum_{j=1}^{n_r}\sum_{k=1}^{N-1}\psi_{k+1}^2(j)} \tag{19}$$

$$J_2(j) = \frac{1}{N-1}\sum_{k=1}^{N-1}(\hat{\phi}_{k+1}(j) - \phi_{k+1}(j))^2 \tag{20}$$

with $\phi_{k+1} = V_1\psi_{k+1}$. Criterion J_2 can be used to construct a two-dimensional graph, taking the spatial location of the j-th component of ϕ_k into account. We will use the notation J_1^{KP} and J_1^{RW} to distinguish between the performance of both prediction models. The same convention is adopted for J_2.

4.1. The breadboard data

The atmospheric turbulence is emulated via a circular plan parallel glass plate mounted on a motor drive unit. Distortions are etched on one side of the glass plate such that it resembles a Kolmogrorov spatial frequency distribution with a $D/r_0 = 5$, where D is the diameter of the telescope and r_0 is Fried's parameter. In this way a single frozen layer disturbance can be emulated. The other main components of the breadboard are a laser source, a Shack-Hartmann sensor, a CCD camera and a data acquisition system. The phase-to-slope matrix G is defined according to the Fried geometry.[1, 2]

The experiment reported here is representative for all experiments conducted. It has considered N equal to 10^4 samples and the sampling frequency equal to 25 Hz. The rotational speed of the glass plate is such that the Greenwood frequency is 4.75Hz. This may seem small but in combination with the rather low sample frequency it gives rise to a considerable temporal error. The number of active spots of the WFS was equal to 50 and therefore $m_s = 100$. The number m_r of observable components of the phase was equal to 77. The results of the experiment are displayed in Figures 1 and 2. In Figure 1 the criterion J_1, defined in (19), is plotted as a function of the order n of the state space model KP in (17) (crosses). When making use of the random walk prediction model in (18), the criterion J_1 equals the dotted line. Since the order of the predictor (18) is $m_r = 77$, we should compare the prediction error obtained with the random walk approach with the performance obtained with an atmospheric disturbance model of order $n = 77$. The ratio of these prediction errors equals $J_1^{KP}/J_1^{RW} = 0.058$. This is corresponds to an improvement of mean square residual phase error by 94%. In Figure 2 we display the criterion J_2 over the aperture to get an idea of the spatial distribution of reduction in prediction error. The prediction was performed using the state space model KP (17) with an order 500. For that order of the predictor (17) the ratio of mean square residual phase is $J_1^{KP}/J_1^{RW} = 2.3 \cdot 10^{-2}$, when averaged over all components of the reconstructed phase. The maximal reduction is achieved in the middle of left front edge and is equal to $4.1 \cdot 10^{-3}$. This is almost equal to the reduction in the center of the WFS. This indicates that for the experimental scenario considered, a more accurate prediction at a particular location can be achieved when use can be made of neighboring measurements.

A final remark is made about the computation time of the novel prediction scheme outlined in section 3.2. The computation time mainly depends on the dimension parameters p, q. Both were considered equal in the calculations. The resulting calculation times on an Intel Pentium IV, 1.70GHz, 512 Mb for different values of the block-Hankel parameter q are displayed in Table 1.

q	time [sec]	q	time [sec]
10	68.03	20	510.28
15	222.28	30	1627

Table 1. Computation time as a function of the block-Hankel size parameter q

4.2. Open-loop data from the William Herschel Telescope

Uncontrolled telescope WFS data from the William Herschel Telescope, situated on La Palma, Canary Islands, have been obtained. The data sets were recorded on June 5 and 8, 1997, in good seeing conditions, 0.5-0.7arcsec

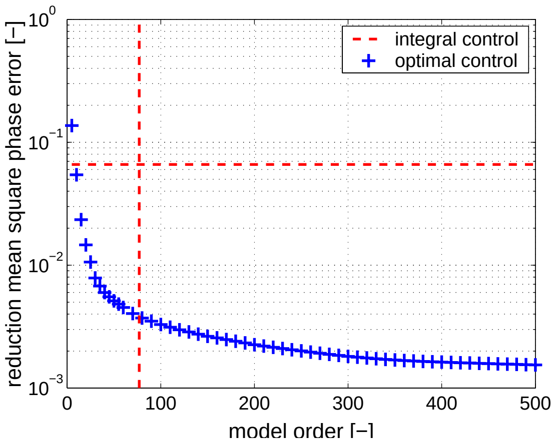

Figure 1. Normalized mean square wavefront prediction error J_1 as a function of the model order. The dashed lines correspond to the performance and equivalent model order of the random walk predictor (18).

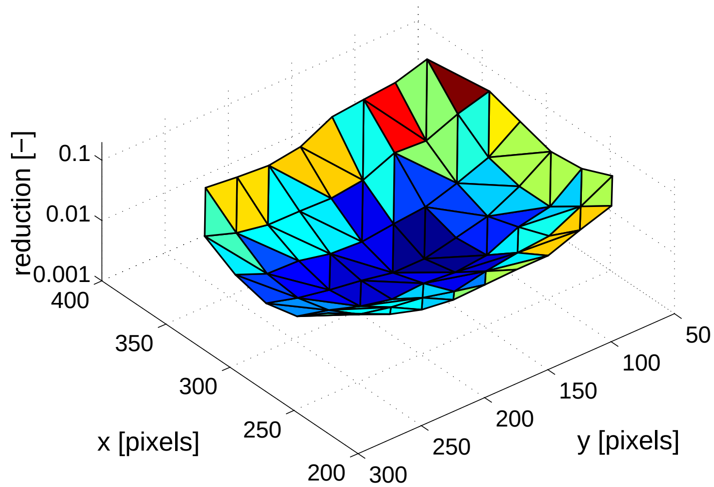

Figure 2. Reduction of the mean square wavefront prediction error $J_2^{KP}(j)/J_2^{RW}(j)$ over the aperture.

FWHM. The instrument used is the JOSE seeing monitor.[17] Its key features are an 8×8 Shack-Hartmann array with 0.5m sub-apertures and 2.4arcsec field of view, imaged onto a 64×64 pixels CCD, resulting in a pixel scale of 0.27arcsec/pixel and it is operated at a wavelength of 700nm. The data sets were taken at a frame rate of 296Hz and each data set contained $N = 10000$ centroid positions on the CCD with an accuracy of approximately 0.1 pixels. Just as in the simulation experiments on the breadboard data, a phase-to-slope matrix G with Fried geometry is used to define the relation between the observable part of the wavefront and the WFS measurement signal. The selected data set contained 48 active spots (i.e. $m_s = 96$ gradient measurements) and the number m_r of observable components of the phase was equal to 67.

Figures 3 and 4 correspond to the results displayed for the simulations on the breadboard data. Just as for the breadboard simulations, the reduction in mean square residual phase error increases with the model order n of the identified disturbance model. The model order needed to outperform the random walk prediction model is much higher than for the breadboard data, but still below the model order of the random walk predictor itself. Since the number of observable components of the phase is $m_r = 67$, the prediction error should be compared with the performance obtained with an atmospheric disturbance model of order $n = 67$. The ratio of the mean square prediction errors for this model order equals $J_1^{KP}/J_1^{RW} = 0.59$, which corresponds to a reduction of 40%. For model orders $n \geq 65$, the state space model KP provides a better prediction than the random walk approach. When the model order n is 500, the ratio of the space and time averaged mean square prediction error obtained by both methods is $J_1^{KP}/J_1^{RW} = 0.18$. Figure 4 provides insight in the spatial distribution of achieved reduction in prediction error. It is interesting to note that the smallest gain in reduction is obtained around the central obscuration of the telescope. This in another indication that the prediction performance can benefit from neighboring channels.

In order to check the validity of the assumption that the observable part of the atmospheric wavefront distortion and the corresponding open-loop WFS signals can be considered as a wide sense stationary over a reasonable amount of time, two wavefront data sets have been considered which have been recorded 3 minutes apart. The first data set has been used to identify a stochastic disturbance model, while the performance of the prediction quality is validated on the second data set. Just as in the previous experiments, the block-Hankel size parameter was chosen equal to $q = 20$ and order of the disturbance model was $n = 256$. This resulted in a reduction of mean square residual prediction error of $J_1^{KP}/J_1^{RW} = 0.1971$ (-7.05dB). The reduction is close to the reduction achieved in the previous experiments, where the same data set is used for identification and validation. This implies that the second order statistics didn't change a lot over a period of three minutes, which demonstrates that the stationarity assumption is reasonable.

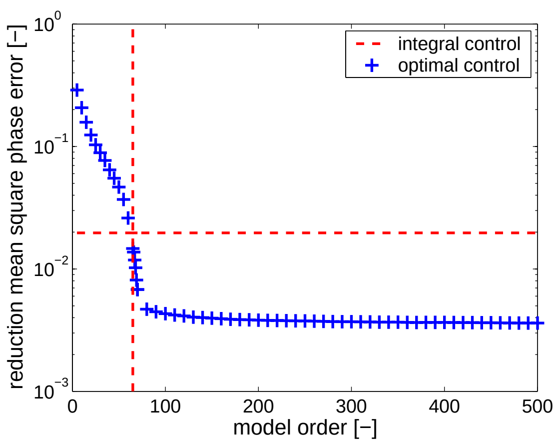

Figure 3. Normalized mean square wavefront prediction error J_1 as a function of the model order. The dashed lines correspond to the performance and equivalent model order of the random walk predictor (18).

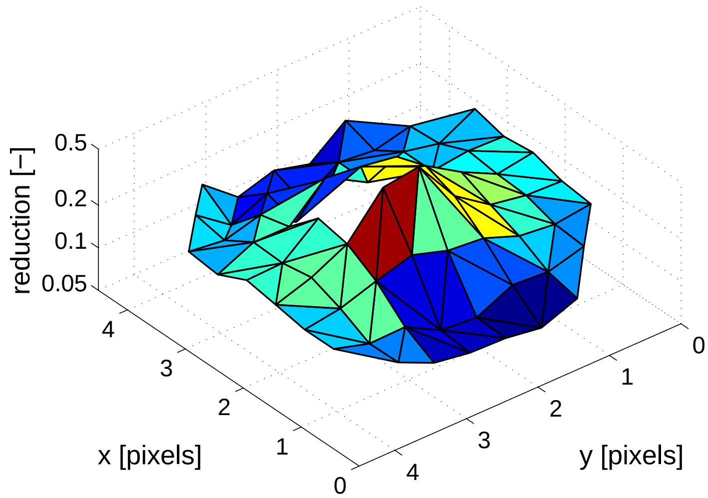

Figure 4. Reduction of the mean square wavefront prediction error $J_2^{KP}(j)/J_2^{RW}(j)$ over the aperture.

5. CONCLUSIONS

This paper outlines a subspace identification algorithm to model the wavefront distortion due to atmospheric turbulence from open-loop wavefront sensor (WFS) measurements. Data-driven modeling has the advantage that it does not depend on discutable assumptions like the assumption of a Kolmogorov spatial distribution or the frozen flow hypothesis. The uncorrected wavefront distortion and the corresponding WFS signal is conceived as a wide sense stationary stochastic process. This implies that the atmospheric disturbance can be modeled as the output of stable linear time invariant (LTI) system with a zero-mean white noise input. Each element of the uncorrected wavefront distortion and WFS signal is represented by a separate output channel of the LTI system. In this way the disturbance model is able to account for the spatio-temporal correlation in the wavefront distortion.

The proposed algorithm explicitly accounts for the fact that only a part of the wavefront can be reconstructed from the open-loop WFS data. By exploiting the singularity in the static relation between WFS measurements and phase, this can even be used to the advantage of the method by reducing the number of channels that has to be modeled. Apart from reducing the size of the identification problem, the reduction leads to an improvement in the numerical conditioning of the calculations. The proposed method yields an atmospheric disturbance model that is in the proper form for control. Indeed, the chosen model structure allows to derive an analytical expression for the optimal controller under certain conditions, as will be outlined in the companion paper.[10] The algorithm has been implemented in a computational efficient way to allow the identification of a disturbance model that describes the relation between all phase points and WFS signals for an AO system of moderate size.

The algorithm has been validated in a simulation study on open-loop data, in which we consider both data obtained from a breadboard setup as well as real-live data from the JOSE seeing-monitor at the William Herschel telescope. In the validation experiments the identified disturbance model is used to predict the wavefront one sample in advance. The prediction capabilities are compared with the random walk predictor which uses the current reconstructed wavefront as the best estimate for the distortion at the next sample. The considered wavefront prediction problem is relevant as it plays in important role in optimal control. The simulation experiments show that a significant reduction in the mean square prediction error can be obtained when compared to the random walk approach. The ratio of mean square prediction errors is 0.057 for the breadboard data and 0.18 for the data obtained from the William Herschel telescope. This corresponds to a reduction of respectively 97.6 and 81.7 percent. Finally the simulations on the JOSE data confirm that the atmospheric wavefront distortion can be considered as wide sense stationary at least over a time scale in the order of a few minutes.

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