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System reliability with correlated components: Accuracy of the Equivalent Planes method

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ABSTRACT

Computing system reliability when system components are correlated presents a challenge because it usually requires solving multi-fold integrals numerically, which is generally infeasible due to the computational cost. In Dutch flood defense reliability modeling, an efficient method for computing the failure probability of a system of correlated components – referred to here as the Equivalent Planes method – was developed and has been applied in national flood risk analysis. The accuracy of the method has never been thoroughly tested, and the method is absent in the literature; this paper addresses both of these shortcomings. The method is described in detail, including an in-depth discussion about the source of error. A suite of system configurations were defined to test the error in the Equivalent Planes method, with a focus on extreme cases to capture the upper bound of the error. The 'exact' system reliability was computed analytically for the special case of equi-correlated components, and otherwise using Monte-Carlo directional sampling. We found that errors in the system failure probability estimates were low for a wide range of system configurations, and became more substantial for large systems with highly-correlated components. In the most extreme cases we tested, the error remained within three times the true failure probability. We provided an example of how one can determine if such error is tolerable in their particular application. We also show the computational advantage of using the Equivalent Planes method; large systems with small failure probabilities which take over 17 h for Monte Carlo directional sampling were computed with the Equivalent Planes in less than one second.

1. Introduction

System reliability analysis investigates the probability that a system will maintain its functionality; that is, the probability that the system will not fail. Computing the failure probability of complex systems, where the components within the system are correlated, usually requires multi-fold integrals, which are generally impossible to evaluate analytically. Consider a vector of random variables, $\mathbf{X} = [x_1, x_2, \ldots, x_n]$, containing both load and strength variables. The failure of the system is represented by the n-fold integral:

$$P_f = \int_{\Omega(\mathbf{X})} f_{\mathbf{X}}(X)dX, \tag{1}$$

where $f_{\mathbf{X}}(X)$ is the multivariate density function of X, and $\Omega(\mathbf{X})$ is the failure space, consisting of all realizations of X that lead to failure of the system. The configuration of the failure space depends on how the components in the system are connected: in series, in parallel, or in some hybrid combination. When connected in series, which is typical in levee systems, $\Omega(\mathbf{X}) = \bigcup_i Z_i(\mathbf{X}) < 0$, where $Z_i(\mathbf{X})$ is the limit state function of the ith component, and where failure of each component is defined by $Z_i(\mathbf{X}) < 0$. Monte Carlo methods to estimate the integral in (1) are typically prohibitively slow, especially in cases where evaluating the limit state functions requires calls to finite element models.

A number of methods have emerged in the past decade to address the need for efficient methods to compute system reliability. Sues and Cesare ([1]) proposed a method (Most Probable Point System Simulation, or MPPSS) in which the reliability of the system components is first computed via a method that returns a closed form of the limit state function (e.g. first- or second-order reliability methods). The limit state functions, together with the Boolean expressions defining failure, are then sampled in a Monte Carlo framework. The authors claim that the size of the system is trivial

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because of the closed form of the limit state functions, but for highly reliable components and/or large systems, it can require billions of samples to acquire the desired accuracy, making this method potentially prohibitively time-consuming. Naess et al. ([2]) proposed a Monte-Carlo-based method in which some tail properties of the distributions are used to substantially improve efficiency. In a follow-up paper ([3]), they tested the method on a large system with thousands of components and found an uncertainty band in which the upper bound is approximately five times the failure probability of the lower bound, for 200,000 samples and a computation time of about 30 min to an hour. The method has not yet been tested on systems in which the limit state function requires calls to an intensive external model (e.g. a finite element model), but will most likely be prohibitively slow given the number of samples required. Kang and Song ([4]) proposed an efficient method (sequential compounding method, or SCM) in which the reliability of the components is first computed, and the components are subsequently combined into equivalent components, two at a time, until the full system reliability is obtained. They tested their method on various system configurations, and found very good accuracy for all the configurations considered in the paper. Chun et al. ([5]) presented a complimentary method to SCM, which computes the sensitivity of the system failure probability to the reliability indexes of the components. The method does not consider the sensitivity of the system failure probability to the random variables that influence the component reliability indexes.

In the Netherlands, the reliability of flood defense systems has been a key research area for decades. Based on a series of papers from the 1980s ([6–9]), an efficient method for combining the failure probabilities of correlated components – referred to here as the Equivalent Planes method – was developed for series systems and implemented in reliability software for the Dutch flood defense system ([10,11]). We want to emphasize that the method was designed for series systems (as flood defense systems are primarily connected in series); two components connected in parallel within a system that is primarily connected in series poses no problem, but the method is not intended for systems of numerous components all connected in parallel. Similar to the MPPSS method of Sues and Casare ([1]), the Equivalent Planes method first computes the failure probability of the components, and then replaces their limit state functions with closed-form expressions for subsequent combining. While the MPPSS method allows generic mathematical formulation, the Equivalent Planes method is restricted to linearized forms of the limit state function (hyperplanes). In contrast to the MPPSS method, the Equivalent Planes method does not rely on Monte Carlo methods. Similar to the Sequential Compounding method from Kang and Song ([4]), the Equivalent Planes method combines components sequentially; they differ most notably in the method to derive the correlation between a combined component and the remaining system components. To accomplish this, the Equivalent Planes method requires information about the autocorrelation of the underlying random variables contributing to failure; the Sequential Compounding method only requires the correlation between components.

The Equivalent Planes method was developed to simultaneously meet two requirements for Dutch flood defense reliability modeling: fast computation for large highly-reliable systems, and the ability to compute influence coefficients of both the random variables and the components. These influence coefficients are critical in Dutch flood defense reliability modeling on two fronts: (1) in deltas, where the flood defense system is subjected to loads fluctuating at different time scales, the influence coefficients are needed to scale the failure probability from the time scale of the highest-fluctuating load to the time scale of interest ([11]), and (2) they give flood defense managers a clear overview which variables, levee segments, or failure mechanisms are contributing the most to the failure probability and require the most attention.

In the Netherlands, the results of the method – the failure probability of a system of flood defenses – have been used in national flood risk analysis, on which major decisions about the safety standards of the defenses have been based ([12–14]). However, the accuracy of the Equivalent Planes method for large systems has never been well investigated. Additionally, although the method is in long-standing use, it remains absent from the literature. This paper serves thus two purposes. The first is to document the method in the literature, and the second is to set up a suite of academic system configurations which we can use to investigate the accuracy of the method.

The paper is laid out as follows. We first describe the Equivalent Planes method in Section 2; we then discuss the source of error in the Equivalent Planes method in Section 3; in Section 4 we describe the various system configurations that we define for investigating error propagation and show the performance of the Equivalent Planes method for these systems; we discuss the idea of tolerable error in Section 5, and close with discussion and conclusions in Section 6.

2. Equivalent Planes method

The Equivalent Planes method computes the failure probability (P_f) of a system of two correlated components, and – by applying it iteratively – the failure probability of a system of any number of components. The ith component is described by a limit state function, Z_i; failure occurs whenever $Z_i < 0$. The method starts with two components, connected in parallel (Eq. (2)) or in series (Eq. (3)). Often these components are correlated; that is, failure of one component will influence the failure probability of the second component.

$$P_f = P(Z_1 < 0 \cap Z_2 < 0) = P(Z_1 < 0) \cdot P(Z_2 < 0 | Z_1 < 0) \tag{2}$$

$$P_f = P(Z_1 < 0 \cup Z_2 < 0) = P(Z_1 < 0) + P(Z_2 < 0) - P(Z_1 < 0 \cap Z_2 < 0) \tag{3}$$

The strategy of the Equivalent Planes method is to replace the conditional probability $P(Z_2 < 0 | Z_1 < 0)$ with an equivalent marginal distribution $P(Z_2' < 0)$ which incorporates the condition $Z_1 < 0$ by having a non-zero density only in the failure space of component 1.

We will describe how the equivalent marginal distribution is computed. But first we will highlight the required information for getting started.

2.1. Getting started

To apply the Equivalent Planes method, we need to know the failure probability of each of the individual components and the correlation between component failures. The latter is driven by common variables. For example, consider a levee section along a river with two failure modes – overtopping and internal erosion; the water level in the river will influence the failure probability of both components, creating correlation between them. To compute the correlation between components, we need information about the variables that cause the correlation: (i) their *autocorrelation* – the correlation between a variable in component 1 and the same variable in component 2 – and (ii) *influence coefficients*, which describe how strongly each variable contributes to failure.

The autocorrelation of the variables can be equal to one in some cases (e.g. variables – like water level – which contribute to different failure modes at the same location will be the same for each failure mode). In other cases (consider soil permeability in two neighboring levee sections), the autocorrelations can be obtained from measurements or from expert opinion, or a combination. To obtain the influence coefficients of the variables, we compute the component failure probabilities using first order reliability method (FORM) (for a description of the FORM method, see [15]). FORM approximates the limit state function as a hyperplane at the design point, with the linearized form shown in Eq. (4); for component i, the coefficients $\alpha_i = [\alpha_{i1}, \dots, \alpha_{in}]$ are the influence coefficients corresponding to a vector of random variables $\mathbf{u} = [u_{i1}, \dots, u_{in}]$; the magnitude of each coefficient indicates the relative influence of each variable on component failure. The random variables are standard normally distributed (they are transformed from their actual marginal distributions via FORM), and the influence coefficients are normalized such that $\sum_{k=1}^{n} \alpha_{ik}^2 = 1$.

$$Z_i = \beta_i - \alpha_{i1}u_{i1} - \alpha_{i2}u_{i2} - \dots - \alpha_{in}u_{in} \tag{4}$$

The component reliability index, β_i, is related to the component failure probability $P_{f,i}$:$\beta_i = \Phi^{-1}(1 - P_{f,i})$, where $\Phi^{-1}(\cdot)$ is the inverse standard normal distribution function.

Once we have the autocorrelations of the variables and the influence coefficients for each component, we calculate the correlation between components according to Eq. (5).

$$\rho(Z_i, Z_j) = \sum_{k=1}^{n} \alpha_{ik} \cdot \alpha_{jk} \cdot \rho_{ijk}, \tag{5}$$

where ρ_{ijk} is the autocorrelation between u_{ik} and u_{jk}. In the remainder of the paper, we use the symbol ρ, without subscripts, to denote the correlation between components; we use the symbol ρ_{ac} to denote the autocorrelation of the variables.

2.2. Failure probability of a two-component system

We start by expressing the limit state function of each component (Z_i) in terms of a single standard normally distributed variable, w_i (Eqs. (6) and (7)). Note that this formulation is equivalent to Eq. (4). For computing the correlation between Z_i and Z_j, we need the individual random variables and their influence coefficients; once we know the correlation, it is more efficient to use the form given in Eqs. (6) and (7).

$$Z_1 = \beta_1 - w_1 \tag{6}$$

$$Z_2 = \beta_2 - w_2 \tag{7}$$

From Eqs. (6) and (7), we can see that, because the reliability index is a constant, the correlation between Z_1 and Z_2 will be the same as the correlation between the variables w_1 and w_2. We therefore write the variable w_2 as a function of w_1 and an independent standard normally distributed variable w_2^* (Eq. (8)), and substitute this expression in the limit state function for the second component (Eq. (9)). This ensures that the correlation between the two components is preserved and ensures that w_2 is still standard normally distributed.

$$w_2 = \rho \cdot w_1 + \sqrt{1 - \rho^2} \cdot w_2^* \tag{8}$$

$$Z_2 = \beta_2 - (\rho \cdot w_1 + \sqrt{1 - \rho^2} \cdot w_2^*) \tag{9}$$

Because of the simplified form of the Z functions, the condition $Z_1 < 0$ is equivalent to $w_1 > \beta_1$ (see Eq. (6)). Therefore, to condition on $Z_1 < 0$ we can simply replace the variable w_1 in Eq. (9) with a new variable w_1', which captures the tail of the w_1 density function above β_1. The density function of w_1', and how it relates to the density function of w_1 is illustrated in Fig. 1.

The expression for Z_2' is thus:

$$Z_2' = \beta_2 - (\rho \cdot w_1' + \sqrt{1 - \rho^2} \cdot w_2^*) \tag{10}$$

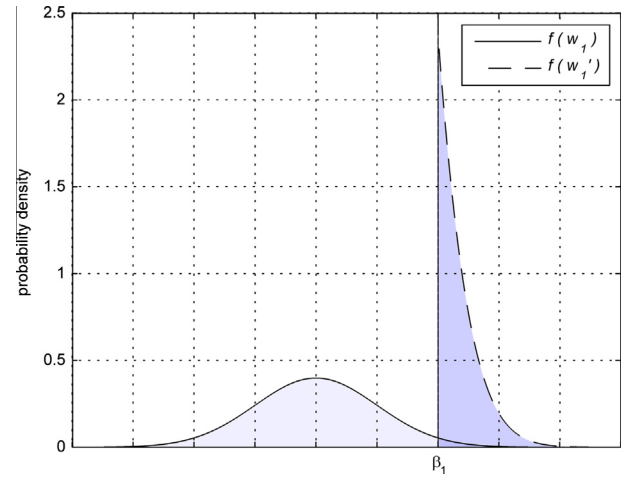

Fig. 1. Density function of w_1 and w_1'.

Note that with Eq. (10), the problem has been reduced from an n-dimensional problem (where n is the number of variables) to a two-dimensional problem, which is why the Equivalent Planes method is so efficient. Computing the marginal distribution $P(Z_2' < 0)$ can be done with any probabilistic technique; we recommend numerical integration because Z_2' is only a function of two variables, so even with very small intervals, numerical integration is efficient and very accurate.

Once we've replaced the conditional distribution $P(Z_2 < 0 | Z_1 < 0)$ with the equivalent marginal distribution $P(Z_2' < 0)$, computing the two-component system failure probability is straightforward; see Eqs. (11) and (12) for the parallel and series systems, respectively.

$$P_f = P(Z_1 < 0) \cdot P(Z_2' < 0) \tag{11}$$

$$P_f = P(Z_1 < 0) + P(Z_2 < 0) - P(Z_1 < 0) \cdot P(Z_2' < 0) \tag{12}$$

With the application of Eq. (11) or (12) we are able to derive the failure probability of a two-component system. The next step is to iterate over this procedure to arrive at the multi-component system reliability.

2.3. Failure probability of an multi-component system

The Equivalent Planes method is iterative, so that once two components have been combined, the two-component system can be considered a new component to combine with a third component, and so on, until all components have been combined.

Computing the correlation between the two-component system and a third component presents a challenge. To compute it via Eq. (5) we will need to know the influence coefficients for the two-component system; that is, we need to represent the two-component system by a linearized limit state function (i.e. a hyperplane). Consider Eq. (4); at the limit state, when $Z_i = 0$, the influence coefficient α_{ik} represents the partial derivative of the reliability index β_i with respect to variable u_{ik}. We can use this to estimate the influence coefficients of the two-component system. In the case where all variables have autocorrelation equal to 1, we obtain the influence coefficients by numerically estimating $\partial\beta_{ij}/\partial u_k$ (where β_{ij} is the reliability index for the system composed of the two components i and j) for each variable u_k. Note that when the autocorrelation is equal to 1, $u_{ik} = u_{jk} = u_k$. When the autocorrelations are not equal to 1, the concept is similar, but the method to compute the influence coefficients is a bit more complex, because the variable u_k is not exactly the same in component i as it is in component j (i.e. $u_{ik} \neq u_{jk}$). Because u_{ik} and u_{jk} are correlated, we can write one as the function of the other, where the function consists of a correlated ($u_{k,c}$) and uncorrelated part ($u_{k,uc}$). We then take the partial derivatives of the system reliability relative to $u_{k,c}$ and $u_{k,uc}$ separately and then combine them as shown in Eq. (13), where α_k^e is the influence coefficient of the kth variable in the equivalent hyperplane for the two-component system.

$$\alpha_k^e = \sqrt{\left(\frac{\partial \beta_{ij}}{\partial u_{k,c}}\right)^2 + \left(\frac{\partial \beta_{ij}}{\partial u_{k,uc}}\right)^2} \tag{13}$$

Once we have computed the influence coefficients for the combined two-component system, we can compute the correlation between it and a third component, and combine these into a three-component system, and so on until we have combined all of the components in our system.

Note that method name – Equivalent Planes – comes from expressing a two-component system as an equivalent hyperplane (of the form in Eq. (4)).

2.4. Practical information

We implemented the methodology described in the preceding sections in Matlab and have made it freely available via Open Earth Tools (https://publicwiki.deltares.nl/display/OET/OpenEarth), which is a repository for free and open source code to handle a variety of problems related to delta and coastal areas ([16]). Open Earth Tools also includes a library of probabilistic tools which are generic and applicable to many problems; the Equivalent Planes algorithm is a part of this library.

3. Error source

The Equivalent Planes method is very efficient, but it comes at a price: it is an approximation. In this section we discuss how the approximation introduces error into the system reliability estimate. In this paper we are focusing on the error incurred using the Equivalent Planes method for combining components with linearized limit state functions. It is important to note that there may also be error introduced in the linearization step; the magnitude of that error is dependent on the behavior of the limit state function, and is not the focus of this paper. For two components with linearized limit state functions in the form of Eq. (4), the Equivalent Planes method is exact; error is introduced when a third component is combined with the equivalent two-component system. Fig. 2[1] illustrates the process by which error is introduced. We begin with a two-component series system (see Fig. 2a), with the failure space defined by the area where $Z_1 < 0 \cup Z_2 < 0$, and the original two-component failure probability $P_f = P(Z_1 < 0 \cup Z_2 < 0)$. After application of the Equivalent Planes method (see Fig. 2b), we have an equivalent limit state function Z_e, and an equivalent failure space defined by $Z_e < 0$. This step is exact which means:

$$P(Z_e < 0) = P(Z_1 < 0 \cup Z_2 < 0) = P_f \tag{14}$$

Fig. 2b shows the trade that was made in failure space; the area A_1 was released in trade for the area A_2 (see also Table 1). This step is exact, so $P(A_1) = P(A_2)$; thus, you can consider this a fair trade. With the introduction of a third component (Fig. 2c), we have a failure space defined by $Z_e < 0 \cup Z_3 < 0$. Fig. 2c shows that the fair trade we had in Fig. 2b is now violated. The area A_3, which represents the portion of A_1 that falls in the failure space of Z_3, represents the error in Fig. 2. We can explain this most clearly as follows.

[1] Note that for legibility, Z_i is denoted Zi for $i = 1, 2, 3, e$ and A_i is denoted Ai for $i = 1, 2, 3, 4, Z3$ in Figs. 2 and 3.

Consider the failure probability of the original two component system, P_f. If we add the third component to the *original* two-component system (Fig. 2a), we are adding the area A_{Z3} (shown in Fig. 2c) to the failure domain. Thus, the system probability becomes:

$$P(Z_1 < 0 \cup Z_2 < 0 \cup Z_3 < 0) = P_f + P(A_{Z3}) \tag{15}$$

If we add the third component to the *equivalent* two-component system (Fig. 2b), we are adding the area A_{Z3} and A_3, and the system probability would be estimated as:

$$P(Z_e < 0 \cup Z_3 < 0) = P_f + P(A_{Z3}) + P(A_3) \tag{16}$$

The error that the Equivalent Planes method makes is thus equal to the difference between Eqs. (15) and (16), which is $P(A_3)$.

Similarly, it can also occur that Z_3 includes some of the gained area (A_2) in its failure space – this area we describe as A_4. This situation is illustrated in Fig. 3 The net error in this case is the probability of A_3 reduced by the probability of A_4.

4. Error under various system configurations

In this section, we investigate the accuracy of the Equivalent Planes-computed system failure probability estimate for various series system configurations. For this, we needed to compute a *reference calculation*; that is, an estimate of the system failure probability that can be considered exact, with which to compare the Equivalent Planes estimate.

4.1. Reference calculation

For systems whose components have equal reliability indexes and are equi-correlated, we were able to compute the exact failure probability of the series system using the formula:

$$P = \int_{-\infty}^{\infty} \left\{1 - \left[1 - \Phi\left(-\frac{\beta_c - v\sqrt{\rho}}{\sqrt{1-\rho}}\right)\right]^m\right\} \varphi(v) dv, \tag{17}$$

where β_c is the reliability index of the components, ρ is the correlation between components, v is a standard normally distributed variable, $\varphi(\cdot)$ is the standard normal density function, and $\Phi(\cdot)$ is the standard normal distribution function.

To compute the 'exact' system failure probability for systems where the components were not equi-correlated, we used a method similar to Sues & Cesare ([1]), only we used Monte Carlo Directional Sampling (MCDS) ([17–19]) instead of crude Monte Carlo, and we implemented a dynamic sample size criterion to ensure a high accuracy (described in Section 4.1.2). We chose MCDS because it is relatively efficient compared with crude Monte Carlo, particularly for the case of linearized limit state functions.

In the following sections we provide a brief explanation of directional sampling (including an efficient approach valid for the case of linearized limit state functions) and the implementation of the dynamic sample size criterion.

4.1.1. Monte Carlo directional sampling

Directional sampling works by sampling directions in the failure space (Eq. (18)), computing the conditional failure probability given the direction (Eq. (19)), and estimating the failure probability as the mean of the conditional probabilities over all N sampled directions (Eq. (20)).

$$\theta = \frac{\mathbf{u}}{\|\mathbf{u}\|} = (\bar{u}_1, \bar{u}_2, ..., \bar{u}_n) \tag{18}$$

$$P_{f,\theta} = \max\{P_i; P_i = P(Z_i < 0|\theta), \quad i = 1...m;\}, \tag{19}$$

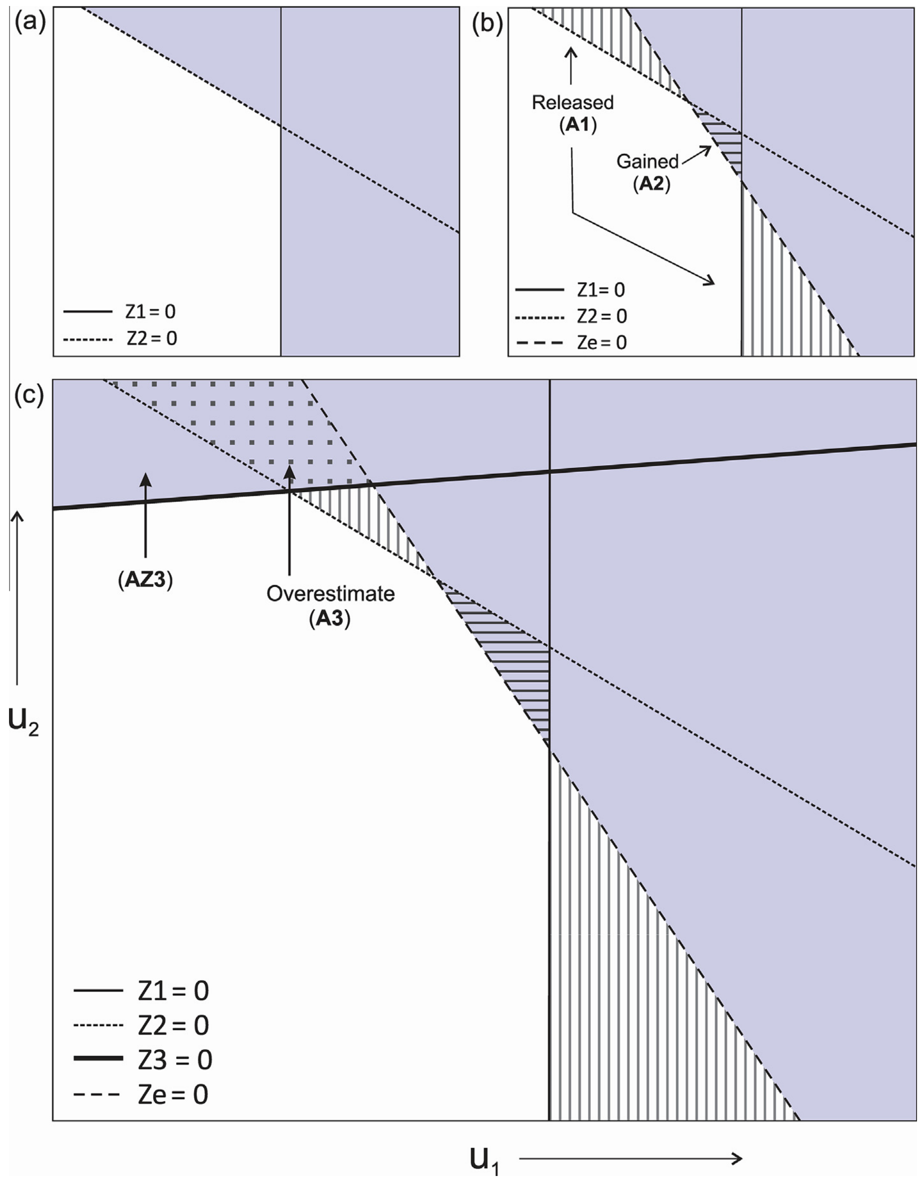

Fig. 2. Introduction of error in the Equivalent Planes method (overestimate). In each subplot, the blue shaded space represents the failure space. The plots show (a) the original failure space of two components with limit state functions Z1 and Z2; (b) situation after application of the Equivalent Planes method – we see the equivalent limit state function Ze and the equivalent failure space; A1 is the released area, and A2 is the area that was gained in trade; (c) situation after the inclusion of a third limit state function Z3 – the shaded area is the failure space of Ze and Z3; A3 is the area that was released in (b) in trade for A2, but is recaptured by Z3, violating the fair trade. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 1
Description of the areas in Fig. 2 and/or Fig. 3.

Area	Description	Z_1/Z_2	Z_e	Z_3
A_1	Failure space released	$Z_1 < 0 \cup Z_2 < 0$	$Z_e > 0$	–[+]
A_2	Failure space gained	$Z_1 > 0 \cap Z_2 > 0$	$Z_e < 0$	–[+]
A_3	Failure space overestimate	$Z_1 < 0 \cup Z_2 < 0$	$Z_e > 0$	$Z_3 < 0$
A_4	Failure space underestimate	$Z_1 > 0 \cap Z_2 > 0$	$Z_e < 0$	$Z_3 < 0$
A_{Z3}	Correct contribution to failure space by Z_3	$Z_1 > 0 \cap Z_2 > 0$	–[*]	$Z_3 < 0$

[+] These steps are prior to the inclusion of the third component.
[*] Not relevant.

where m is the number of components in the system.

$$\hat{P}_f = \frac{1}{N}\sum_{j=1}^{N} P_{f,\theta}(j) \tag{20}$$

Each direction is defined by a unit vector θ in the standardized normal space; see Eq. (18). We obtain this unit vector by first sampling all of the n standard normally distributed variables. The vector from the origin to the sampled point in the n -dimensional variable space gives us the vector $\mathbf{u}$ in Eq. (18). Normalizing this vector gives us the directional unit vector θ.

In general, Eq. (19) can be cumbersome to compute because it requires searching in an n-dimensional space for the limit state nearest to the origin (in the given direction). However, we can exploit the linearity of the limit state functions in our case to simplify this process. Let λ be the distance from the origin to the nearest limit state function; then the vector giving the direction and the distance to failure is $\lambda\theta = (\lambda\bar{u}_1, \lambda\bar{u}_2, ..., \lambda\bar{u}_n)$. Because our limit state functions are linear we know that $Z = \beta - \alpha_1 u_1 - ... - \alpha_n u_n$, so we can plug in the failure vector $(\lambda\bar{u}_1, \lambda\bar{u}_2, ..., \lambda\bar{u}_n)$ and solve for λ by setting $Z = 0$.

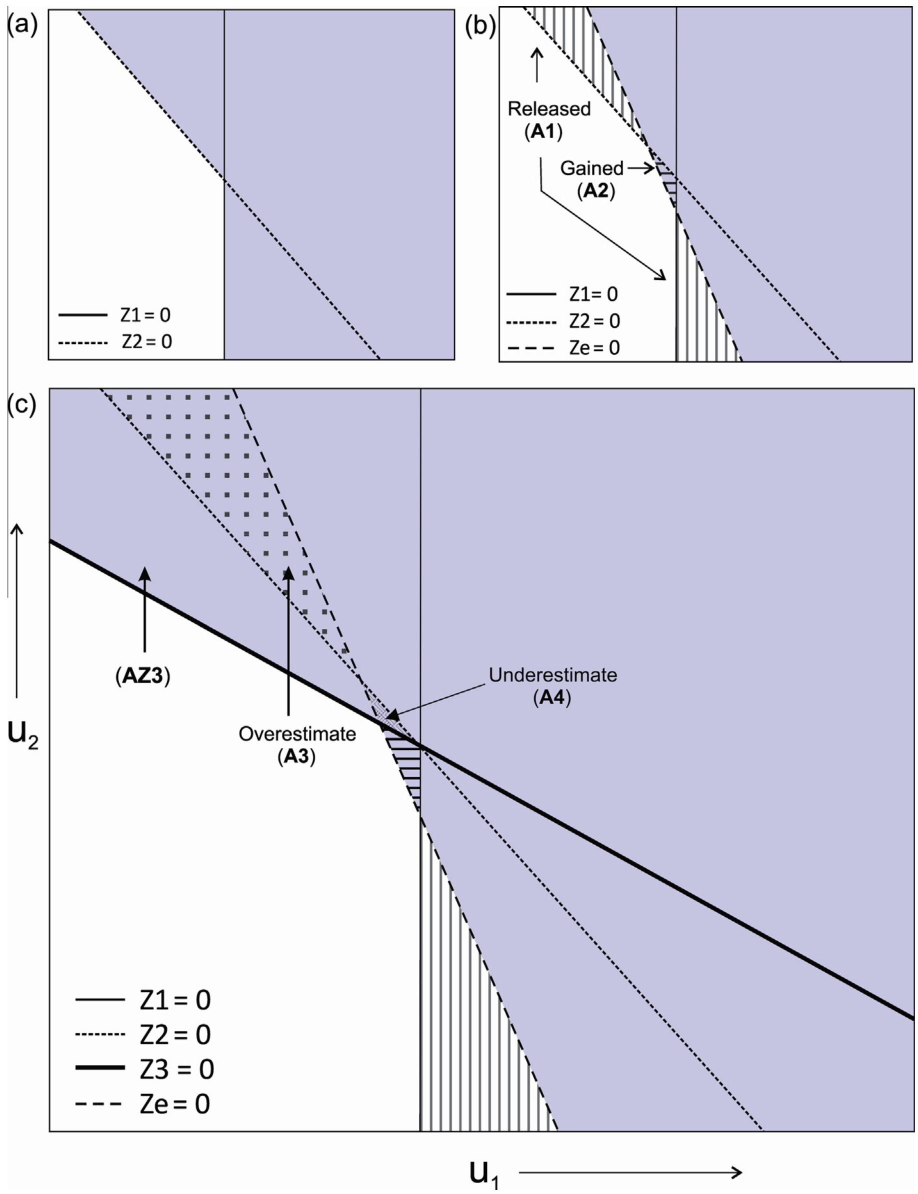

Fig. 3. Introduction of error in the Equivalent Planes method (over- or underestimate). In each subplot, the blue shaded space represents the failure space. The subplots show (a) the original failure space of two components; (b) situation after application of the Equivalent Planes method – we see the equivalent limit state function Ze and the equivalent failure space; A1 is the released area, A2 is the gained area; (c) situation after the inclusion of a third limit state function Z3 – the shaded area is the failure space of Ze and Z3; A3 is the area released in (b) in trade for A2, but is recaptured by Z3, the area A4 was gained in (b) but is already part of the system failure due to Z3. A3 and A4 represent a violation of the fair trade in (b). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\beta = \alpha_1(\lambda\bar{u}_1) + \ldots + \alpha_n(\lambda\bar{u}_n) = \lambda\sum_{i=1}^{n}\alpha_i\bar{u}_i \Rightarrow \lambda = \frac{\beta}{\sum_{i=1}^{n}\alpha_i\bar{u}_i} \tag{21}$$

We compute λ using Eq. (21) for each limit state function in our system and take the minimum as our distance to failure. We then use the Chi Squared distribution to compute the conditional probability given the direction and the squared distance to failure ([19]).

4.1.2. Sample size criterion

Our criterion for when the sample size was large enough was a 95% confidence that the difference between the MCDS system reliability estimate ($\hat{\beta}$) and the true value (β) is less than a defined value C:

$$P(|\hat{\beta} - \beta| < C) = 95\% \tag{22}$$

The important consideration when choosing a value for C in Eq. (22) is that it should be small relative to the errors in the Equivalent Planes method, or relative to errors that would be considered important. We chose a value $C = 0.01$, which we felt was a good compromise between efficiency (not requiring too many samples) and having an error that was small relative to anything we would be concerned about in practice.

The implementation of the stop criterion is described by the flow chart in Fig. 4. The standard deviation of the failure probability estimate $\hat{P}_f$ is computed as follows:

$$\hat{\sigma}_{P_f} = \sqrt{\frac{1}{N\cdot(N-1)}\cdot\sum_{i=1}^{N}\left(P_{f,\theta}(i) - \hat{P}_f\right)^2} \tag{23}$$

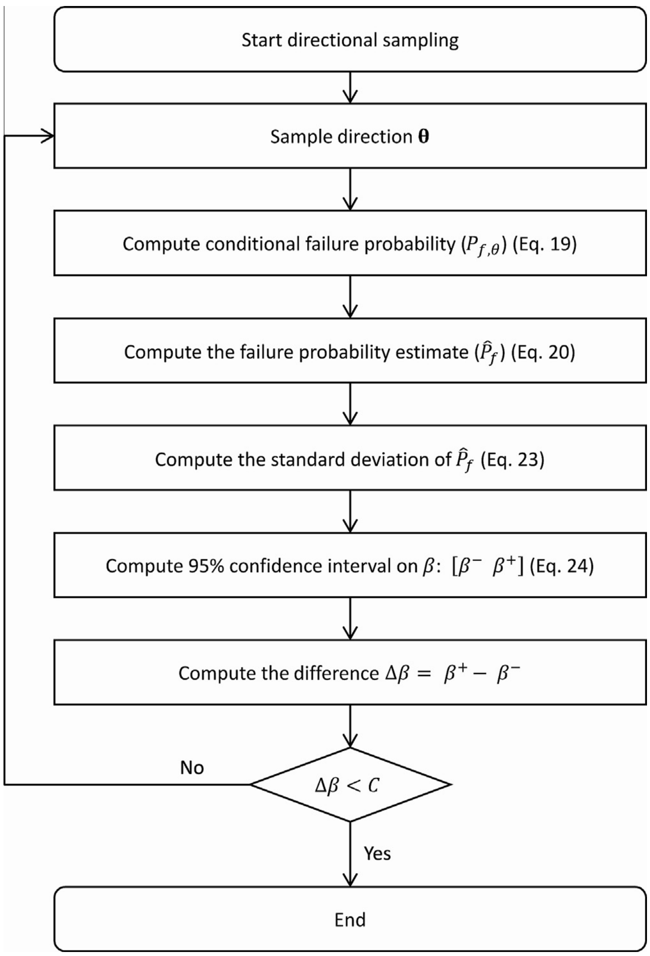

Fig. 4. Flow chart describing the stop criterion for directional sampling.

The 95% confidence interval on the reliability estimate $\hat{\beta}$ is described by the lower and upper bound on the interval (β^- and β^+, respectively):

$$\beta^- = -\Phi^{-1}(\hat{P}_f + 2\hat{\sigma}_{P_f}); \quad \beta^+ = -\Phi^{-1}(\hat{P}_f - 2\hat{\sigma}_{P_f}), \tag{24}$$

where Φ^{-1} is the inverse standard normal distribution function.

4.2. System configurations

We wanted to test various system configurations to explore under which conditions the accuracy of the Equivalent Planes may become a problem. We defined a system configuration based on the following parameters:

(i) n – number of variables.
(ii) m – number of components.
(iii) ρ – correlation between components.
(iv) β_c – component reliability index.
(v) ρ_{ac} – auto-correlation of the variables.

To test the influence of these parameters we considered a number of system configurations, which are described in the following sections. For each system configuration we computed the 'exact' failure probability (P_f) using either Eq. (17) for equi-correlated components, or Monte Carlo directional sampling otherwise. We then compared the exact system failure probability with the Equivalent Planes-estimated system failure probability ($\hat{P}_f$).

The system configurations we chose were based on a number of considerations. First, we wanted to test extreme system configurations, in an effort to compute bounds on the error in the Equivalent Planes method. To this end, for all the systems we considered, we kept the component reliability indexes equal across all the components. We did this because if we had allowed the components of a system to have different reliabilities, the smallest component reliability would dominate the system reliability, and would make the errors caused by combining the more reliable components negligible. Second, we only considered series systems, because the Equivalent Planes method was specifically designed with levee systems in mind, which are predominantly series systems. Inclusion of two components connected in parallel within a predominantly series system should not impact the error. However, the method was not designed to compute large parallel systems, and hence we did not consider such systems. Third, we most extensively considered cases with equi-correlated components, because the exact solution can be computed analytically (see Eq. (17)). This allowed us to investigate large systems with high reliability indexes which would have been too computationally intensive to compute with the Monte Carlo directional sampling method. We investigated cases where the components were not equally correlated, but not extensively.

4.2.1. Case I: equal correlation between components

Case I investigates series systems with equally reliable components, and equal correlation between all components. We enforced the correlation between components as follows. Assume m limit state functions, each of which is a function of n variables. For a desired correlation between components (ρ), we set the influence coefficients of the variables equal for all m limit state functions ($\alpha_{1i} = \alpha_{2i} = \ldots = \alpha_{mi}; \quad i = 1 \ldots n$), and set the autocorrelation of all of the random variables equal to ρ. Eq. (5) then reduces to:

$$\rho(Z_i, Z_j) = \sum_{k=1}^{n} \alpha_{ik} \cdot \alpha_{jk} \cdot \rho_{ijk} = \rho \sum_{k=1}^{n} \alpha_k^2 = \rho \tag{25}$$

We also varied the number of components (m), and the reliability index of the components (β_c). We fixed the number of variables to three ($n = 3$). Note that because the variables are only partially autocorrelated, the dimensionality of the problem is much higher than 3 (see Eq. (8)); in fact the dimensionality will be equal to the product of n and m (the number of variables and the number of components). Table 2 summarizes the system configurations we considered for the case of equal correlations.

We computed the exact system reliability using Eq. (17), but we also ran the Monte Carlo directional sampling for several of the cases, to assess the computation time involved. In typical cases, Eq. (17) cannot be used, and it is then useful to compare the computation time of the Monte Carlo procedure with the Equivalent Planes method. The computation times are presented in Section 6.

The results are presented in Figs. 5–8, one figure for each of the correlations 0.2, 0.5, 0.7, and 0.9. It is clear that the Equivalent Planes method performs best for components with high reliability indexes, and where the correlation between components is not too high. Fig. 9 highlights the relationship between the error in the Equivalent Planes method and the correlation between the components; the error is given as a factor difference in the failure probability, which is the ratio of the Equivalent Planes-computed failure probability to the exact failure probability. Table 3 shows the factor difference in the failure probability for systems with 250 components. It shows that in the worst case the system failure probability estimated with Equivalent Planes is 2.5 times the exact system failure probability. In the best cases, they are equal. Table 4 shows the error in the Equivalent Planes estimate of the reliability indexes. The conclusions are the same as for Table 3, but viewing the error

Table 2
Parameters defining the system configurations for the Case I.

Variable	Value(s)
Number of variables (n)	3
Number of components (m)	3–250
Component reliability (β_c)	3, 4, 5, 6
Autocorrelation (ρ_{ac})	ρ
Correlation between components (ρ)	0.1–0.9 (in increments of 0.1), 0.95, 0.99

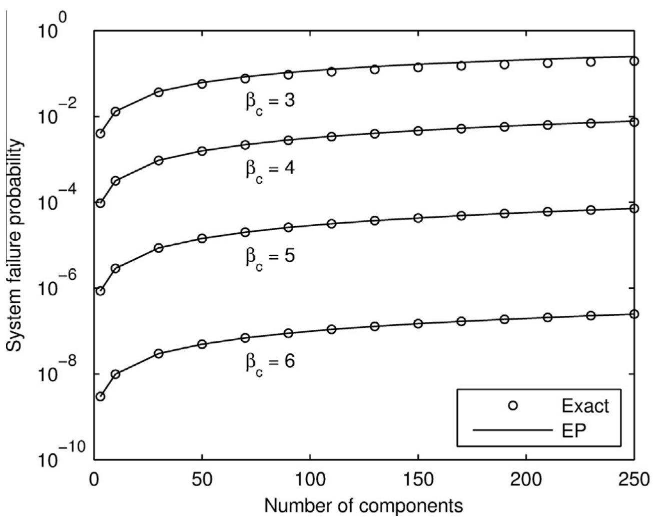

Fig. 5. Performance of the Equivalent Planes (EP) method for series systems with 3–250 components, all equi-correlated with correlation coefficient 0.2.

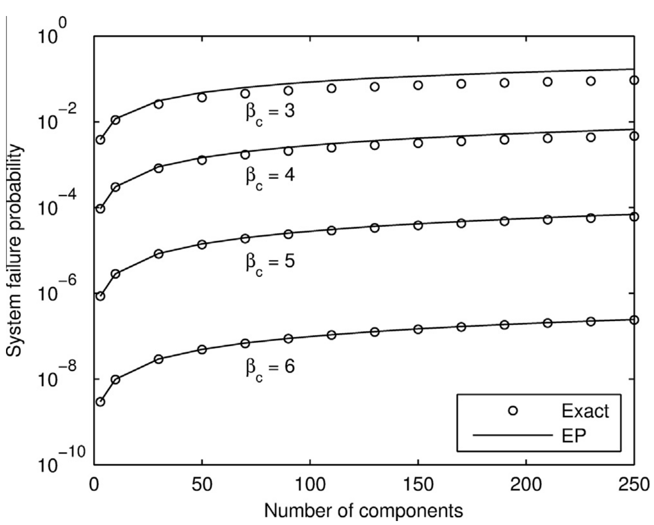

Fig. 6. Performance of the Equivalent Planes (EP) method for series systems with 3–250 components, all equi-correlated with correlation coefficient 0.5.

in terms of reliability index will be useful when we give an example of how to assess tolerable error in Section 5.1.

4.2.2. Case II: unequal correlation between components

Case II investigates a series system of components with equal reliability indexes but unequal correlation coefficients. The correlation between component i and component j in an m–component system was computed according to Eq. (26), as in [4]; this formulation ensures the correlation matrix is positive definite.

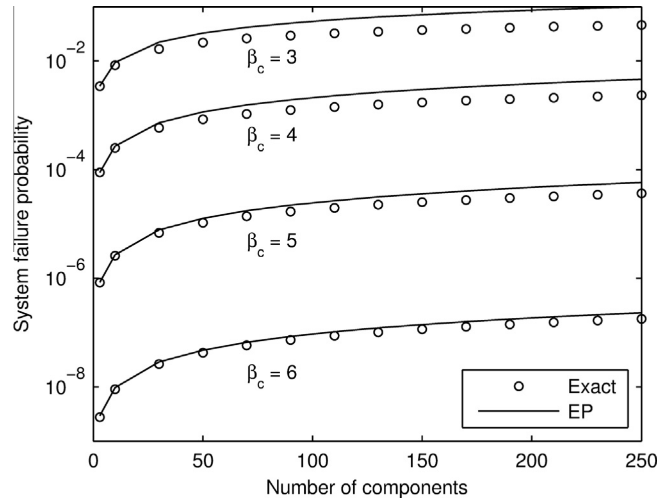

Fig. 7. Performance of the Equivalent Planes (EP) method for series systems with 3–250 components, all equi-correlated with correlation coefficient 0.7.

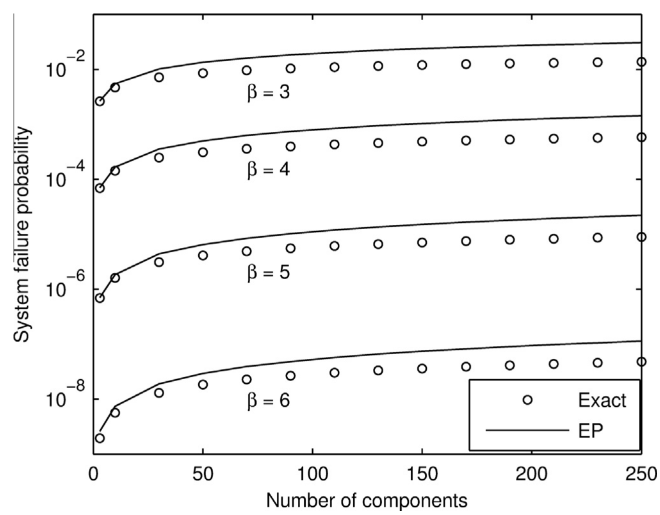

Fig. 8. Performance of the Equivalent Planes (EP) method for series systems with 3–250 components, all equi-correlated with correlation coefficient 0.9.

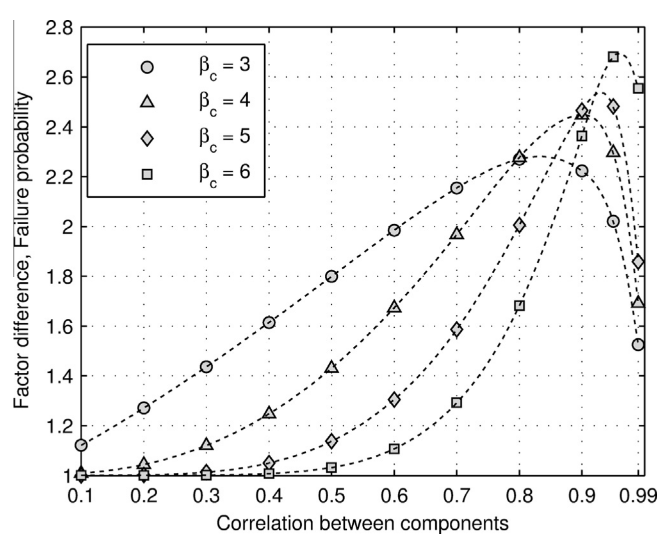

Fig. 9. Factor difference in system failure probabilities (ratio of Equivalent Planes-computed to exact) for series systems with 250 components, for correlation between components ranging from 0.1 to 0.99.

Table 3
Factor difference in the system failure probabilities (ratio of Equivalent Planes-computed to exact) for the configurations in Case I, for a system with 250 components.

	$\rho = 0.2$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\beta_c = 3$	1.3	1.8	2.2	2.2
$\beta_c = 4$	1.0	1.4	2.0	2.4
$\beta_c = 5$	1.0	1.1	1.6	2.5
$\beta_c = 6$	1.0	1.0	1.3	2.4

Table 4
Error in the Equivalent Planes reliability indexes for the configurations in Case I (difference between Equivalent-Planes computed and exact), for a system with 250 components.

	$\rho = 0.2$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
$\beta_c = 3$	−0.18	−0.36	−0.40	−0.33
$\beta_c = 4$	−0.02	−0.13	−0.22	−0.26
$\beta_c = 5$	0.00	−0.03	−0.11	−0.20
$\beta_c = 6$	0.00	−0.01	−0.05	−0.16

$$\rho_{ij} = 1 - \frac{|i-j|}{m-1}, \quad i,j = 1,\ldots,m \tag{26}$$

Enforcing the correlation structure given by Eq. (26) requires choosing the right mixture of influence coefficients and autocorrelations for the random variables in the limit state functions, given the constraint in Eq. (5). We set the number of variables in each limit state function equal to m (the number of components), and set the autocorrelation of each variable equal to 1. This reduces the relationship between the influence coefficients and the correlation matrix (see Eq. (5)) to:

$$\rho = \alpha^T \alpha \tag{27}$$

The advantage of having the correlation matrix in the form of Eq. (27) is that we can easily derive the influence coefficients for any positive-definite correlation matrix using Cholesky decomposition.

$$\alpha = \mathrm{CHOL}(\rho) \tag{28}$$

We computed the 'exact' system reliability using Monte Carlo directional sampling. We considered 5-, 10-, and 50-component systems, and component reliability indexes of 3, 4, and 5. We did not consider systems larger than 50 components, because of the computational cost of the Monte Carlo simulations. For example, for 50 components and a component reliability index of 5, we needed $1.3 * 10^8$ samples for the Monte Carlo directional sampling to converge (see sample size criterion in Section 4.1.2). The details of this case are given in Table 5.

The results are shown in Fig. 10. We find very good agreement between the Equivalent Planes-computed system failure probability and the Monte-Carlo-computed system failure probability.

Table 5
Parameters defining the system configurations for Case II.

Variable	Value(s)
Number of variables (n)	$n = m$
Number of components (m)	5, 10, 50
Component reliability (β_c)	3, 4, 5
Autocorrelation (ρ_{ac})	1
Correlation between components (ρ)	See Eq. (26)

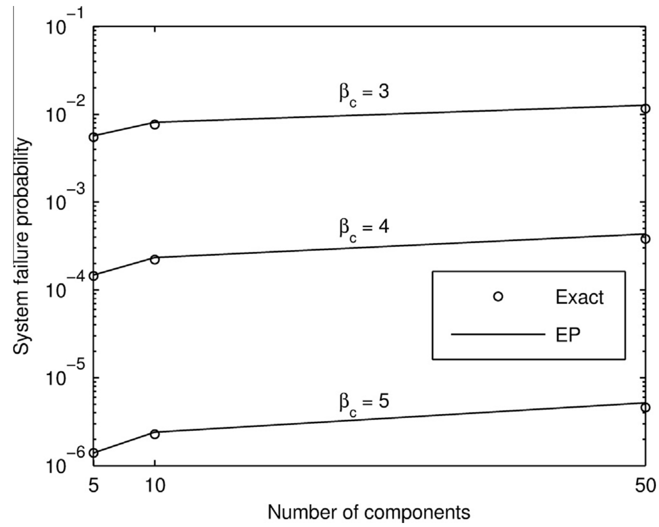

Fig. 10. Performance of the Equivalent Planes (EP) method for series systems with 5, 10 and 50 components, correlated according to the correlation structure defined in Eq. (26).

4.2.3. Case III: limit states which span all directions

This case investigates an extreme situation in which the directions of the linearized limit state functions of the components span a three-dimensional space. For many components – all with equal component reliability indexes – this begins to enclose a spherical safe region. This is a very unrealistic situation, but is useful for testing how the method performs under such extremes.

To generate the limit state functions, we generated directional normal vectors (perpendicular to the limit state hyperplane) by using an three-dimensional integer-based grid. Essentially we chose a maximum integer, x_{max}, and constructed a grid with points placed at all integers from $-x_{max}$ to x_{max}. We then connected each point to the origin, normalized these vectors, and removed any duplicates. Duplicates arise when the line between the origin and multiple grid points share the same angle (e.g. consider a two-dimensional grid with the points [1,1] and [2,2]). The number of limit state functions (i.e. the number of components m) is then a function of the total number of integers (which is equal to

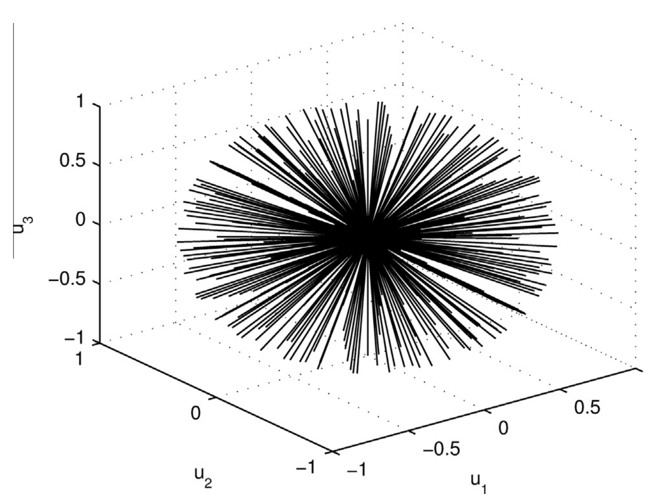

Fig. 11. Directional normal vectors defining the limit state functions which densely span a 3-dimensional space.

$2x_{\max} + 1$), the number of dimensions n, and the number of duplicates d:

$$m = (2x_{\max} + 1)^n - d \tag{29}$$

An example of a set of directional normal vectors is illustrated for three dimensions in Fig. 11.

Because the number of components grows exponentially with the number of variables, we restricted this case to three dimensions.

We chose an autocorrelation equal to 1 for all of the variables to limit the dimensionality of the case, in order to make the reference (Monte Carlo) computations more efficient. The correlations between components (which are variable in this case) were not explicitly chosen. Table 6 summarizes the system configurations we considered.

Table 6
Parameters defining the system configurations for the case spanning an n-dimensional space.

Variable	Value(s)
Number of variables (n)	3
Max integer ($x_{\max}$)	1, 2, 3, 4
Component reliability (β_c)	3, 4, 5, 6
Autocorrelation (ρ_{ac})	1

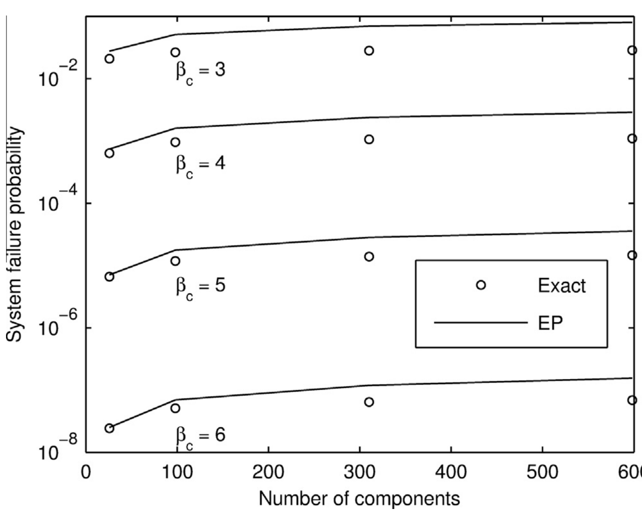

Fig. 12. Performance of the Equivalent Planes (EP) method for series systems with components whose linearized limit state functions span a 3-dimensional space, with unequal correlation coefficients between components, for 26, 98, 310, and 598 components.

Table 7
Factor difference in the system failure probabilities (ratio of Equivalent Planes-computed to exact), for the configurations in Case III.

	$m = 26$	$m = 98$	$m = 310$	$m = 598$
$\beta = 3$	1.3	1.9	2.5	2.8
$\beta = 4$	1.2	1.7	2.2	2.6
$\beta = 5$	1.1	1.5	2.0	2.4
$\beta = 6$	1.0	1.4	1.9	2.2

Table 8
Error in the Equivalent Planes system reliability indexes (difference between Equivalent Planes-computed and exact), for the configurations in Case III.

	$m = 26$	$m = 98$	$m = 310$	$m = 598$
$\beta = 3$	−0.12	−0.30	−0.43	−0.49
$\beta = 4$	−0.05	−0.16	−0.25	−0.30
$\beta = 5$	−0.02	−0.09	−0.17	−0.21
$\beta = 6$	−0.01	−0.06	−0.11	−0.15

The results of Case III are presented visually in Fig. 12. Table 7 shows the factor difference in the system failure probability, and Table 8 shows the error in the system reliability index. The worst case tested – 598 components and component reliability indexes of 3 – results in an Equivalent Planes system failure probability estimate that is 2.8 times the exact (Monte-Carlo computed) system failure probability. In the best case – 26 components and component reliability indexes of 6 – the errors in the Equivalent Planes method are negligible.

4.2.4. Case IV: uncorrelated components

This case investigates the extreme situation where all components are uncorrelated. This situation is easy to compute analytically (Eq. (2) reduces to the product of the component probabilities). It is relevant to test how the method performs under this extreme, since some cases might be nearly uncorrelated in practice. The analytical solution was used for comparison of the results.

To set up the uncorrelated case, we set the number of (independent) random variables equal to the number of components ($n = m$), where each component depends on only one of the variables, which will have an influence coefficient of 1 (see Eq. (4)). Each limit state function is written in terms of all of the variables, but all but one of the influence coefficients will be zero. We chose the number of components, which then determines the number of variables. We also chose the component reliability indexes. The value of the autocorrelation is irrelevant, because each variable appears in only one limit state function. Table 9 summarizes the system configurations we considered for the case of uncorrelated components.

The results show that the Equivalent Planes method is practically exact for uncorrelated components (Fig. 13).

Table 9
Parameters defining the system configurations for the case of uncorrelated components.

Variable	Value(s)
Number of variables (n)	m
Number of components (m)	3–250
Component reliability (β_c)	3, 4, 5, 6
Autocorrelation (ρ_{ac})	Irrelevant (each variable appears in only one component)
Correlation between components (ρ)	0

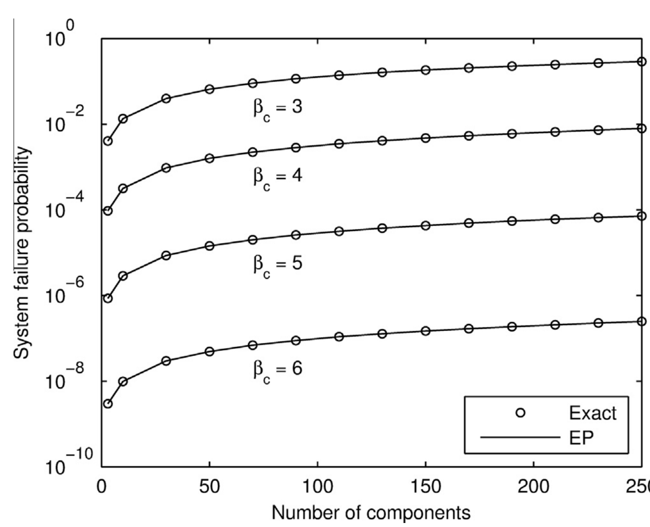

Fig. 13. Performance of the Equivalent Planes (EP) method for series systems with 3–250 uncorrelated components.

5. Acceptable error

This section focuses on translating the error in the Equivalent Planes method to real, tangible terms (e.g. costs), so that we can determine how much error is acceptable. This translation is application-dependent, but we provide an example here for the case of levee systems, principally for illustration purposes. Each specific application will need its own case-specific analysis, but the approach should be similar.

5.1. Example: levee systems

We considered the application to levee systems and provide a simple example of how errors in system reliability can be translated to impacts on levee design, and further to costs. The idea of this section is not to provide full rigor in determining the impact on levee design and costs, but to obtain an order-of-magnitude estimate that can help us decide if the error in our estimate is tolerable.

We considered the failure mechanism overflow, which essentially considers whether the crest height of the levee is high enough to hold back extreme water levels. The limit state function for overflow is the difference between the levee height (H) and the water level at the levee (W).

$$Z = H - W \tag{30}$$

We assigned a Type I generalized extreme value distribution to the water level, with a scale parameter of 0.28 and a location parameter of 2.6, which corresponds to 1/100, 1/1000, and 1/10,000 year water levels of 3.9, 4.5, and 5.2 m, respectively. The system reliability index, β, informs us of the failure probability, or equivalently (in this case) the probability that the water level is higher than the levee height.

$$P_f = \Phi(-\beta) = P(H - W < 0) = P(W > H) \tag{31}$$

Because we know the distribution of the water level F_w, we can determine the design height – that is, the levee height that corresponds to the reliability index β:

$$H = F_W^{-1}(1 - \Phi(-\beta)) \tag{32}$$

We can use Eq. (32) to translate errors in the system reliability to errors in the design levee height. We considered system reliability indexes $\beta = \{3, 4, 5, 6\}$, and errors in the system reliability $\varepsilon = \{0.01, 0.05, 0.1, 0.2, 0.5\}$, and computed the difference in design levee height between the erroneous and true system reliabilities; the results are presented in Table 10.

To translate the design height differences into costs, we used cost curves derived as part of a national cost-benefit analysis of flood protection measures in the Netherlands ([14,20,21]). We considered the extreme cases of a very rural levee and a very urban levee. A very rural levee is one in which the area surrounding the levee is undeveloped, and thus allows for easy expansion of the levee base when the levee is heightened; this ensures that the slope of the levee does not become too steep. Such an expansion will be impossible for a very urban levee because the surrounding area is already fully developed. In such cases, retaining walls are often required to compensate for the steep slope resulting from the levee heightening. Furthermore, in the urban case, a road and a bike lane are also typically present on the levee. For both the rural and urban case there is a base cost – that is, a portion of the cost that is height-independent. In the rural case, this is the cost of removing and replacing the levee revetment; in the urban case, it is the cost of the retaining wall and the road and bike lane. Fig. 14 shows example cost curves for rural and urban levees. The costs are expressed per km of levee, and as a function of the required levee height increase.

Table 10
Difference in design levee height (in meters) due to errors in the system reliability estimate, for different system reliability indexes (β) and different error magnitudes (ε).

	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.5$
$\beta = 3$	0.01	0.05	0.09	0.19	0.48
$\beta = 4$	0.01	0.06	0.12	0.24	0.61
$\beta = 5$	0.01	0.07	0.14	0.29	0.75
$\beta = 6$	0.02	0.09	0.17	0.34	0.88

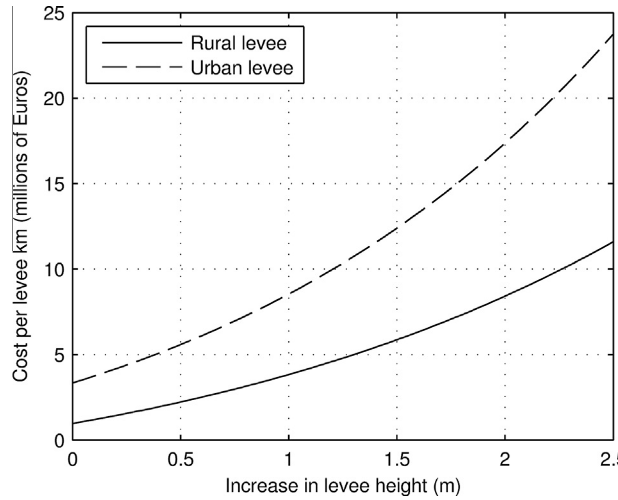

Fig. 14. Cost curve for levee heightening, for a rural and urban levee, based on data for Dutch levees.

To use the cost curve (Fig. 14) to translate the error in design height to costs, we must consider the specific case at hand. For example, suppose we have a situation where our current levee must be raised by 1 m to satisfy a required reliability; however, the error in our reliability estimate leads us to believe it must be raised by 1.5 m. Then we would read from the curve the difference in costs between 1 and 1.5 m: approximately 2 million Euros per km of levee for the rural case, or about 4 million Euros per km for the urban case. Whether this error is tolerable depends on a number of factors that are case-specific. Note that the converse situation – that our error would be an underestimate of the levee design height – would result in a less expensive improvement measure, but a higher risk. The costs associated with the increased risk are more complex to assess than the costs of an improvement measure.

6. Discussion and conclusion

The Equivalent Planes method has been used in Dutch system reliability modeling of flood defenses for decades. The reliability model is at the heart of national flood risk analysis, the results of which are used to drive major flood prevention policies in the Netherlands. The critical role of the model motivated this research to determine the accuracy of the Equivalent Planes method, and under which situations we may encounter unacceptable error.

We used Monte Carlo directional sampling method to compute 'exact' reliability estimates with which to compare the Equivalent Planes results. Table 11 shows the computation times[2] (in minutes) required for Monte Carlo directional sampling (with an imposed accuracy of 0.01 on the estimate of the system reliability index),

[2] Computation times are based on a 2.8 GHz computer with 8 GB RAM.

Table 11
Computation time (in minutes) for Monte Carlo directional sampling to compute the system reliability, for series systems of $m = 3$, 10, 50, 100, and 250 components with equal component reliability indexes (β_c) and equally correlated with a correlation coefficient of 0.9.

	$m = 3$	$m = 10$	$m = 50$	$m = 100$	$m = 250$
$\beta_c = 3$	<1	<1	1	1	3
$\beta_c = 4$	<1	1	7	13	32
$\beta_c = 5$	<1	8	188	424	1079

for some of the system configurations that we tested as part of Case I (see Section 4.2.1), for an inter-component correlation of 0.9. The Equivalent Planes method computed each of these configurations in less than 1 s. Consider a system with 250 components and component reliability indexes of 5; in this case, the Monte Carlo directional sampling method required 1079 min – over 17 h – compared with 0.95 s required by the Equivalent Planes method. In reality, systems are likely to have a large number of components, and component reliability indexes that are greater than 5; thus, the reduction in computation time for real systems will be substantial.

We computed the error in the Equivalent Planes method for different system configurations, and found that when the components are not too correlated (e.g. a correlation coefficient up to about 0.5), the error in the method is generally negligible, particularly when the components have high reliability indexes. Inaccuracies become apparent for large systems with highly correlated components, and for components with lower reliability indexes. In all cases, the Equivalent Planes system failure probability estimates were within a factor of three times the correct system failure probability. Recall that we are investigating upper bounds on the error; these results are for extreme system configurations in which the components all have equal reliability indexes. In reality, a few components will likely dominate the failure probability, and the error will be much lower. Furthermore, even three times the correct failure probability can be quite negligible for systems with very small failure probabilities. For example, consider a system of 250 equi-correlated components, with component reliability indexes of 6, correlated with a coefficient of 0.9; the true failure probability is 4.81E−8 and the estimate is 1.14E−7, for a factor difference of 2.4 (see Table 3). In many applications, where the probability needs to be below a certain safety standard, this difference will not be important. Furthermore, other uncertainties in the reliability analysis – for example, due to the parameterization of the random variables contributing to failure – will likely overshadow this small error, making it essentially negligible.

We discussed how to evaluate if the error is tolerable, and provided an example for a levee system with loads similar to those found in the river regions of the Netherlands. It is important that researchers investigate tolerable error for their specific case to determine if the Equivalent Planes method will be sufficiently accurate. When it is, it is a very attractive method, particularly when considering the gain in computational time over more exact methods.

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