Event-based State Estimation with Negative Information

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***Abstract*— To reduce the amount of data transfer in networked systems, measurements are usually taken only when an event occurs rather than periodically in time. However, this complicates estimation problems considerably as it is not guaranteed that new sensor measurements will be sampled. In order to cope with such event sampled measurements, an existing state estimator is modified so that any divergent behavior in estimation results will be curtailed. To start, a general formulation of event sampling is proposed, which is then used to set up a state estimator combining stochastic as well as set-membership measurement information according to a hybrid update: when an event occurs the estimated state is updated using the stochastic measurement received (positive information), while at periodic time instants no new measurement is received (negative information) and the update is based on knowledge that the sensor value lies within a bounded subset of the measurement space. An illustrative example further shows that the developed estimator has an improved representation of estimation errors compared to purely stochastic estimators for various event sampling strategies.**

I. Introduction

Event-based state estimation is an emerging alternative to classical, time-periodic estimators with many relevant applications in networked systems. In contrast to sampling periodically in time, event-based estimators employ an *event* sampling strategy for triggering new measurements at instants of well designed events. Two examples of event sampling are “Send-on-Delta” [1], [2] and “Matched Sampling” [3] as illustrated in Section III. Resource limitations of networked systems in communication and energy are among the main motivations for pursuing event-based estimation, since the event sampling strategy employed aims to reduce the amount of measurements exchanged. The additional value of event sampling is best noticed in networked systems with (simplistic) sensors capturing continuous or high frequency signal information, e.g., temperature and position sensors. The approach is unfavorable in systems capturing complex, asynchronous sensory data, such as object detection with cameras. Notice further that employing event sampling will result in some extra computational power at the sensor and that, although measurements arrive at instants of an event, estimation results are usually required periodically in time. For example, serving as input to a monitoring system or a periodic controller as it is depicted in the networked system of Figure 1.

Therefore, the main issue in event-based state estimation is finding a suitable approach for processing *event* sampled measurements while keeping stable estimation results *periodically* in time. Asynchronous estimation approaches can be employed for solving this issue. Yet, it was shown in [4] that they will have unstable behavior as merely sampled measurements taken at the events are exploited for an estimation update, while it is unknown whether new measurement will be sampled at all. Moreover, additional measurement information is available in between two events, since criteria for triggering the next event can be turned into a set-membership property of the sensor value when no new measurement is sampled (*negative information*)[1]. Several event-based state-estimators have been proposed that exploit this additional set-membership information in a state update, yielding stable estimation results for various event sampling strategies. See, for example, the estimators proposed in [5], [6]. The main drawback of both approaches is that they treat the set-membership measurement information as a stochastic measurement and thereby, introduce systematic approximations and incorrect models of the estimation error. This issue can be solved by a set-membership estimation approach instead of a stochastic one, e.g., [7]–[9]. However, drawbacks of these approaches is their computational complexity and the fact that sensor noise typically follows a stochastic model.

To solve these issues, the estimation problem addressed in this article employs a stochastic representation for modeling noises and a set-membership representation for incorporating additional measurement information derived from event sampling criteria. In particular, the goal of this article is

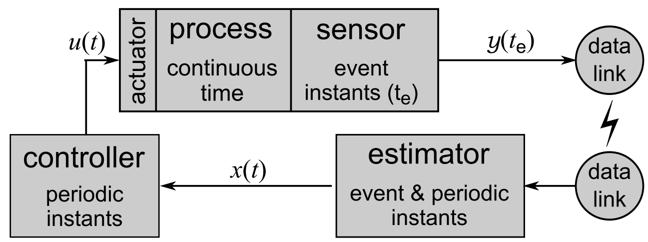

Fig. 1. The considered networked system including an event-based estimator and a time-periodic controller. Therein, $y(t_e)$ denotes a measurement taken at an event instant t_e, $x(t)$ is the state at either an event instant or at a time-periodic instant, and $u(t)$ is the control action.

[1]Assuming that sensor information is captured continuously or at high rates.

to modify the combined stochastic and set-membership state estimator proposed in [10], such that *any* type of event sampling strategy is exploited for periodic estimation results. To that extent, a mathematical description of event sampling is introduced. This forms the basis for setting up a stable event-based state-estimator according to a hybrid update. More precisely, when an event occurs, the estimated state is updated using the received measurement value, while at periodic time instants the update is based on the inherent knowledge that the sensor value lies within a bounded set used to define the event. This bounded set and the hybrid update are key for achieving asymptotic bounds on the *modeled* estimation error. An illustrative case study further demonstrates that the proposed event-based estimator has more realistic results on modeled estimation errors compared to existing alternatives for various event sampling strategies.

II. Preliminaries

$\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ define the set of real numbers, non-negative real numbers, integer number, and non-negative integer numbers, respectively. Further, $\mathbb{Z}_\mathbb{C} := \mathbb{Z} \cap \mathbb{C}$, for some $\mathbb{C} \subset \mathbb{R}$. The null-matrix and identity-matrix of corresponding dimensions are denoted as 0 and I, respectively. For a time-varying signal $x(t) \in \mathbb{R}^n$, let $x(t_\mathsf{e})$ denote the value of x at the e-th sampling instant $t_\mathsf{e} \in \mathbb{R}_+$. A transition-matrix $A_{t_2-t_1} \in \mathbb{R}^{m\times n}$ is defined to relate a vector $u(t_1) \in \mathbb{R}^m$ to a vector $x(t_2) \in \mathbb{R}^n$, i.e., $x(t_2) = A_{t_2-t_1}u(t_1)$. The q-th element of a vector $x \in \mathbb{R}^n$ is denoted with $[x]_q$. The transpose, trace and inverse (when it exists) of a matrix $A \in \mathbb{R}^{n\times n}$ are denoted as $A^\top$, tr(A), A^{-1}, respectively. The 2-norm of a vector $x \in \mathbb{R}^n$ is denoted as $\|x\|_2$. Given that $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are positive definite, denoted with $A \succ 0$ and $B \succ 0$, then $A \succ B$ denotes $A - B \succ 0$. Further, $A \succeq 0$ denotes A is positive semi-definite.

The probability density function (PDF) of a random vector $x \in \mathbb{R}^n$ is denoted as $p(x)$ and the conditional PDF of x given $y \in \mathbb{R}^q$ is denoted as $p(x|y)$. Further, $\mathbb{E}[x]$ and $cov(x)$ denote the expectation and covariance of x, respectively. If x is Gaussian distributed, denoted as $p(x) = \mathcal{G}(x,\mu,P)$, then $\mu := \mathbb{E}[x]$ and $P := cov(x)$. Any $\mathcal{G}(x,\hat{x},P)$ can be represented by its unitary sub-level-set $\mathbb{L}_{\hat{x},P} \subset \mathbb{R}^n$, yielding an ellipsoidal set defined as $\mathbb{L}_{\hat{x},P} := \{x | (x-\hat{x})^\top P^{-1}(x-\hat{x}) \leq 1\}$.

III. Event sampling for state estimation

Consider a perturbed dynamic process described by a generic discrete-time state-space model, for some $A_\tau \in \mathbb{R}^{n\times n}$, $B_\tau \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and sampling interval $\tau \in \mathbb{R}_+$, i.e.,

$$x(t) = A_\tau x(t-\tau) + B_\tau w(t-\tau) \ , \tag{1a}$$

$$y(t) = Cx(t) + v(t) \ . \tag{1b}$$

The state-vector $x \in \mathbb{R}^n$ is affected by the process noise $w \in \mathbb{R}^m$ and the measurement $y \in \mathbb{R}^l$ is subject to sensor noise $v \in \mathbb{R}^l$. Both noise distributions are Gaussian, i.e.,

$$p(w(t)) := \mathcal{G}(w(t),0,W) \text{ and } p(v(t)) := \mathcal{G}(v(t),0,V) \ ,$$

for some *given* $W \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{l\times l}$. Basically, the system description in (1a) could be perceived as a discretized version of a continuous-time process $\dot{x}(t) = Fx(t) + Gw(t)$, where $A_\tau := e^{F\tau}$ and $B_\tau := (\int_0^\tau e^{F\eta}\mathrm{d}\eta)G$.

The sensor employs an event sampling strategy for obtaining the e-th event sampled measurement denoted as $y(t_\mathsf{e})$. Measurements are then used for estimating the state x. Usually, an estimate of x is required periodically in time rather than at the instants of an event, e.g., as input to a monitoring system or a periodic controller. To accommodate both events as well as periodic instants, let us define the two sets $\mathbb{T}_e \subset \mathbb{R}_+$ and $\mathbb{T}_p \subset \mathbb{R}_+$ corresponding to all events and all periodic instants, respectively. If $\tau_s \in \mathbb{R}_+$ denotes the sampling time, then

$$\mathbb{T}_e := \{t_\mathsf{e} \mid \mathsf{e} \in \mathbb{Z}_+\} \quad \text{and} \quad \mathbb{T}_p := \{c\tau_s \mid c \in \mathbb{Z}_+\} \ .$$

The events t_e are generated according to a particular triggering criterion defined by the employed event sampling strategy. A detailed account on such triggering criteria for various event sampling strategies is the topic of the next section, followed by a section for deriving a set-membership property $y(t) \in \mathbb{H}(\mathsf{e},t)$ on the sensor value in between two events e and $\mathsf{e}-1$, for some $\mathbb{H}(\mathsf{e},t) \subset \mathbb{R}^l$. This latter property is important as it gives additional sensor information when no new measurement has been received and thus can be used for improving and updating estimation results. See Figure 2 for a schematic setup of such an event-based state estimator exploiting the fact that not receiving a new measurement (*negative information*) still induces valuable information for a state update, i.e., information that the measured value $y(t)$ is bounded by a known set.

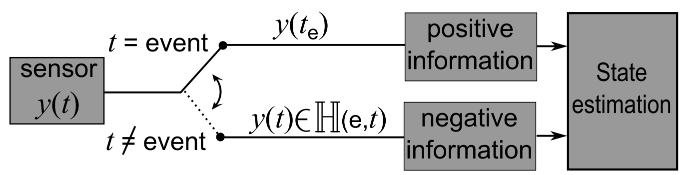

Fig. 2. A schematic setup of the developed estimator, where measurement information is dependent on the type of sample instant: at an event a new measurement $y(t_\mathsf{e})$ is received, while in between the two events e and $\mathsf{e}-1$ one has the knowledge that the sensor value lies within a bounded set $\mathbb{H}(\mathsf{e},t)$.

Let us continue with some illustrative approaches for sampling and how triggering criteria in case of event-based sampling can be turned into a set-membership property.

A. Illustrative Approaches for Sampling

A straightforward approach is sampling periodically in time, though aperiodic strategies have recently emerged as a viable alternative. Event-based sampling is such an alternative, for which some examples are Send-on-Delta and Matched Sampling, as proposed in [1], [2], [11] and in [3], [12], respectively. Typically, event sampling defines that triggering a next event $\mathsf{e} \in \mathbb{Z}_+$ depends on the current sensor value $y(t) \in \mathbb{R}^l$ and previously sampled measurements $y(t_{\mathsf{e}-c})$, for all $c < \mathsf{e}$. Let us present the event sampling strategies Send-on-Delta and Matched Sampling in more detail, see also Figure 3 and Figure 4, before deriving a crucial property of (most) event sampling strategies exploited by event-based state-estimators.

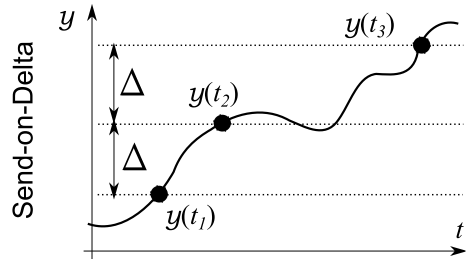

Fig. 3. An illustration of the triggering criteria for the event sampling strategy Send-on-Delta. Note that criteria for the Send-on-Delta approach depend on the previous measurement sample and some threshold $\Delta \in \mathbb{R}_+$.

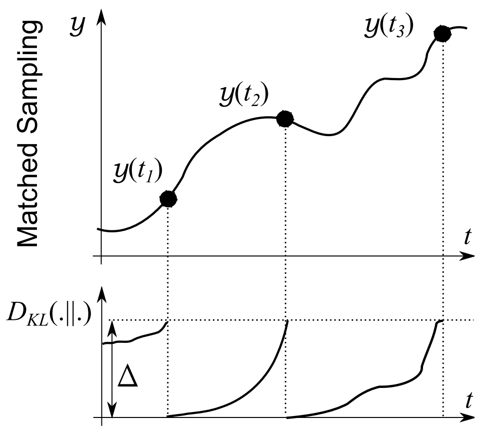

Fig. 4. An illustration of the triggering criteria for the event sampling strategy Matched Sampling. Note that criteria for the Matched Sampling approach depend on the Kullback-Leibler divergence $D_{KL}(\cdot||\cdot)$ and a threshold $\Delta \in \mathbb{R}_+$.

1) Send-on-Delta: The event sampling strategy Send-on-Delta, proposed in [1], [2], [11], defines the criterion for triggering a next event t_{e}, as follows:

$$t_{\mathsf{e}} = \inf\{t > t_{\mathsf{e}-1} \mid \|y(t) - y(t_{\mathsf{e}-1})\|_2 > \Delta\} \ , \quad \text{for some } \Delta > 0 \ .$$

Suppose that the previously sample measurement was $y(t_{\mathsf{e}-1}) = 3$ and that $\Delta = 1$. Then, the next event instant for taking a new measurement sample occurs when the sensor value $y(t)$ either crosses 2 or 4. As such, with Send-on-Delta it is guaranteed that the current sensor value $y(t)$ lies within a set characterized by $\pm\Delta$ around the previously sampled $y(t_{\mathsf{e}-1})$. This further implies that the event triggering criteria of Send-on-Delta can be rewritten by introducing a subset $\mathbb{H}(\mathsf{e},t) \subset \mathbb{R}^l$ in the measurement space, i.e.,

$$t_{\mathsf{e}} = \inf\{t > t_{\mathsf{e}-1} \mid y(t) \notin \mathbb{H}(\mathsf{e},t)\} \ , \tag{2}$$

$$\mathbb{H}(\mathsf{e},t) = \left\{ y \in \mathbb{R}^l \mid \|y - y(t_{\mathsf{e}-1})\|_2 \leq \Delta \right\} \ . \tag{3}$$

2) Matched Sampling: The event sampling strategy Matched Sampling, proposed in [3], [12], defines the criterion for triggering a next event t_{e} as

$$t_{\mathsf{e}} = \inf\left\{ t > t_{\mathsf{e}-1} \mid D_{\mathrm{KL}}\big(p_1(x(t))||p_2(x(t))\big) > \Delta \right\} \ ,$$

for some $\Delta > 0$. Herein, $D_{\mathrm{KL}}(\cdot||\cdot) \in \mathbb{R}_+$ denotes the Kullback-Leibler divergence of the PDFs $p_1(x(t))$ and $p_2(x(t))$. The PDF $p_2(x(t))$ represents a prediction of $x(t)$ based on the estimation result at $t_{\mathsf{e}-1}$, while $p_1(x(t))$ corresponds to an estimate of $x(t)$ that is obtained by updating the prediction $p_2(x(t))$ with the current sensor value $y(t)$. Suppose that the sensor employs an asynchronous Kalman filter to estimate $x(t)$ and let the estimation results at the previous sample instant $t_{\mathsf{e}-1}$ be characterized by the Gaussian PDF $\mathcal{G}(x(t_{\mathsf{e}-1}), \hat{x}(t_{\mathsf{e}-1}), P(t_{\mathsf{e}-1}))$. Then, for a given interval $\tau := t - t_{\mathsf{e}-1}$, the predicted $p_2(x(t))$ and updated $p_1(x(t))$, yields

$$\begin{aligned} p_1(x(t)) &:= \mathcal{G}\big(x(t), \theta_1(t), \Theta_1(t)\big) \ , \\ p_2(x(t)) &:= \mathcal{G}\big(x(t), \theta_2(t), \Theta_2(t)\big) \ , \end{aligned} \tag{4}$$

where,

$$\begin{aligned} \Theta_2(t) &:= A_\tau P(t_{\mathsf{e}-1}) A_\tau^\top + B_\tau W B_\tau^\top \ , \\ \theta_2(t) &:= A_\tau \hat{x}(t_{\mathsf{e}-1}) \ , \\ \Theta_1(t) &:= \big(\Theta_2^{-1}(t) + C^\top V^{-1} C\big)^{-1} \ , \\ \theta_1(t) &:= \Theta_1(t)\big(\Theta_2^{-1}(t)\theta_2(t) + C^\top V^{-1} y(t)\big) \ . \end{aligned}$$

An explicit expression of the Kullback-Leibler divergence for the two Gaussian PDFs $p_1(x(t))$ and $p_2(x(t))$ of (4) was derived in [13], i.e.,

$$\begin{aligned} & D_{\mathrm{KL}}\big(p_1(x(t))||p_2(x(t))\big) \\ & \qquad := \alpha(t) + \frac{1}{2}\big(\theta_1(t) - \theta_2(t)\big)^\top \Theta_2^{-1}(t)\big(\theta_1(t) - \theta_2(t)\big) \ , \\ & \alpha(t) := \frac{1}{2}\Big(\log|\Theta_2(t)|\,|\Theta_1(t)|^{-1} + \mathrm{tr}\big(\Theta_2^{-1}(t)\Theta_1(t)\big) - l\Big) \ . \end{aligned}$$

In addition, note that the event triggering criteria for Matched Sampling can be rewritten into a criterion similar to (2) by redefining the set $\mathbb{H}(\mathsf{e},t)$. A characterization of this set was already derived in [3], [12] as the ellipsoidal set

$$\begin{aligned} \mathbb{H}(\mathsf{e},t) &= \mathbb{L}_{C\theta_2(t), \Phi(t)} \ , \\ \Phi(t) &:= 2\big(\Delta - \alpha(t)\big)\big(V^{-1} C \Theta_1(t) \Theta_2^{-1}(t) \Theta_1(t) C^\top V^{-1}\big)^{-1} \ . \end{aligned} \tag{5}$$

In fact, a generalization of triggering criteria employed by any event sampling strategy is introduced in the next section based on this measurement "event" set $\mathbb{H}(\mathsf{e},t)$.

B. A Set-membership Property for Event Sampling

The above examples indicate that triggering the e-th event sample is based on the current sensor value $y(t)$ and a bounded Borel set $\mathbb{H}(\mathsf{e},t) \in \mathbb{R}^l$ in the measurement space. The latter set denotes a collection of all allowable values that $y(t)$ may take at any $t > t_{\mathsf{e}-1}$. As such, generating the next event instant t_{e}, given $t_{\mathsf{e}-1}$, can be generalized as

$$t_{\mathsf{e}} := \inf\{t > t_{\mathsf{e}-1} \mid y(t) \notin \mathbb{H}(\mathsf{e},t)\} \ . \tag{6}$$

Further, requiring $y(t_{\mathsf{e}-1}) \in \mathrm{int}(\mathbb{H}(\mathsf{e},t))$ ensures that $t_{\mathsf{e}} > t_{\mathsf{e}-1}$, i.e., two consecutive events are not simultaneously triggered. An example of $\mathbb{H}[\mathsf{e},t]$ is depicted in Figure 5. Note that Figure 5 as well as the illustrative sampling approaches presented in Section III-A demonstrate that (most) event sampling strategies satisfy the following property, see also [14]:

Proposition III.1 *Let $y(t)$ be sampled with an event strategy similar to* (6). *Then, $y(t) \in \mathbb{H}(\mathsf{e},t)$ holds for any $t \in [t_{\mathsf{e}-1}, t_{\mathsf{e}})$.*

Proposition III.1 formalizes the inherent measurement knowledge of event sampling, i.e., *not* receiving a new measurement for any $t \in [t_{\mathsf{e}-1}, t_{\mathsf{e}})$ implies that $y(t)$ is included in $\mathbb{H}(\mathsf{e},t)$. Furthermore, since the employed event sampling strategy is given, one can derive a characterization of $\mathbb{H}(\mathsf{e},t)$ prior to the event instant t_{e}. Yet, it should be noted that different event sampling strategies will result in different measurement "event" sets $\mathbb{H}(\mathsf{e},t)$. Nonetheless, it is exactly this set-membership property of event sampling that gives the additional measurement information used to perform a state update not only at instants of an event but also periodically in time when no new measurement is received.

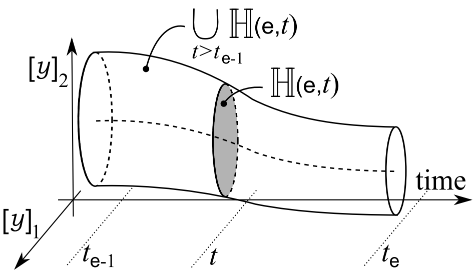

Fig. 5. An example of $\mathbb{H}[\mathsf{e},t] \subseteq \mathbb{R}^2$ in the *measurement-space* at time $t \geq t_{e-1}$, where $[y]_q$ denotes the q-th element of $y \in \mathbb{R}^2$. In addition, the collection of event sets for all $t > t_{e-1}$, i.e., $\cup_{t>t_{e-1}} \mathbb{H}(\mathsf{e},t)$, is illustrated as well.

IV. The estimation problem and existing solutions

Before alternative solutions for processing event sampled measurements are presented, let us state the estimation problem in more details. The estimator should be able to exploit the event sampled measurement, from any event sampling strategy, so that estimation results can be computed periodically. This means that the sample instants of the estimator are the combination of both event and periodic time instants, for which $\mathbb{T} := \mathbb{T}_e \cup \mathbb{T}_p$ is introduced. The main challenge is to cope with the hybrid nature of measurement information available, i.e.,

- at the instants $t \in \mathbb{T}_e$ of an event, a new measurement value $y(t) = Cx(t) + v(t)$ is received;
- at periodic time instants $t \in \mathbb{T}_p$, the measurement information is a property that the sensor value lies within a bounded subset, i.e., $y(t) \in \mathbb{H}(\mathsf{e},t)$ see Proposition III.1.

Any estimator developed should thus be able to incorporate both stochastic and set-membership measurement information. This further implies that, apart from an estimated mean $\hat{x}(t) \in \mathbb{R}^n$, estimation results will contain a stochastic part characterized by an error-covariance $P(t) \in \mathbb{R}^{n\times n}$ and a set-membership part characterized by the error-set $\mathbb{X}(t) \subset \mathbb{R}^n$.

Existing solutions related to asynchronous estimation, e.g., [7], [9], [15], perform a prediction of $x(t)$ at periodic time instants $t \in \mathbb{T}_p$ when no new measurement is received. These solutions either select a stochastic representation for their estimation results or a set-membership characterization. Furthermore, it was shown in [4] that a diverging behavior of the error-covariance $P(t)$ will occur (or of the error-set $\mathbb{X}(t)$). To curtail this diverging behavior, one should exploit the inherent measurement information provided by Proposition III.1 and *update* the estimation results at *all* sample instants $t \in \mathbb{T}$. Although an existing solution that employs this idea was proposed in [5], the considered (stochastic) estimator is especially designed for the event sampling strategy Send-on-Delta and scalar measurements. An extension towards any event sampling strategy has been proposed in [6]. Yet, the estimator turns the set-membership measurement information into a stochastic approximation of the sensor value. As such, a pure derivation of a combined stochastic and set-membership state estimator suitable for event sampling is still missing.

V. An event-based state estimator with negative information

The preceding considerations unveiled that a desired estimator takes account for the hybrid nature of event-triggered and time-periodic measurement information. In order to handle measurements affected by stochastic sensor noise, while at the same time exploit the set-membership property of Proposition III.1, the sensor model in (1b) is extended to an implied measurement $z(t)$, for some ellipsoidal set $\mathbb{L}_{0,E(t)} \in \mathbb{R}^l$, i.e.,

$$\begin{aligned} &z(t) = Cx(t) + v(t) + e(t) \\ &\text{where } v(t) = \mathcal{G}(v(t), 0, V) \text{ and } e(t) \in \mathbb{L}_{0,E(t)}. \end{aligned} \tag{7}$$

Besides the sensor noise $v(t)$, an unknown but bounded error $e(t)$ has been introduced to capture the set-membership property explained in Proposition III.1 (present at periodic time instants). The expression in (7) indicates that the sensor noise and bounded error are characterized by a covariance matrix $V \in \mathbb{R}^{l\times l}$ (see Section III) and an ellipsoidal shape matrix $E \in \mathbb{R}^{l\times l}$, respectively. It is important that this implied measurement is equivalent to model (1b) at the instant of events, i.e., $z(t) = y(t)$ and thus $e(t) \in \emptyset$ for all $t \in \mathbb{T}_e$, while at periodic time instants $t \in \mathbb{T}_p$ the variables $Cx(t)$ and $e(t) \notin \emptyset$ will follow a characterization depending on the employed event sampling strategy so to exploit "negative information", i.e., that the sensor still provides information even when no event is triggered. Note that at these periodic time instants the noise covariance V remains the same as in the original sensor model (1b). In contrast, specific values of the ellipsoidal shape matrix $E \succ 0$, modeling $e(t) \in \mathbb{L}_{0,E(t)} \notin \emptyset$, should be derived from the current event triggering criterion, which is illustrated later in this section. First, the estimator shall be explained.

The underlying idea is to compute a state estimate $\hat{x}(t)$ that simply minimizes the (maximum possible) mean squared error (MSE) in the presence of both stochastic and set-membership uncertainties. The developed estimator, which is referred to as EBSE-NI (Event-Based State Estimator with Negative Information), is based on the approach presented in [10]. The error associated to the state estimate $\hat{x}(t)$ is composed of a stochastic and a set-membership component, for some covariance matrix $P(t) \in \mathbb{R}^n$ and error-set $\mathbb{X}(t) \subset \mathbb{R}^n$, i.e.,

$$\begin{aligned}&\hat{x}(t)-x(t)=\tilde{x}^{\text{stoc}}(t)+\tilde{x}^{\text{set}}(t)\ ,\\ &\text{such that } p(\tilde{x}^{\text{stoc}}(t))=\mathcal{G}\big(\tilde{x}^{\text{stoc}}(t),0,P(t)\big)\\ &\text{and}\qquad \tilde{x}^{\text{set}}(t)\in\mathbb{X}(t).\end{aligned}$$

In line with the introduced error variable $e(t)$ in (7), let $\mathbb{X}(t)$ be ellipsoid, i.e., $\mathbb{X}(t) := \mathbb{L}_{0,X(t)}$ and is assumed to be centered at zero for some ellipsoidal shape matrix $X(t)\succeq 0$. Further, since the set-membership error is non-stochastic and independent from stochastic errors, the according MSE yields

$$\begin{aligned}&\mathbb{E}\big[\big(\hat{x}(t)-x(t)\big)^\top\big(\hat{x}(t)-x(t)\big)\big]\\ &=\underbrace{\mathbb{E}\big[\big(\tilde{x}^{\text{stoc}}(t)\big)^\top\big(\tilde{x}^{\text{stoc}}(t)\big)\big]}_{=\text{trace}\left(P(t)\right)}+\underbrace{\big(\tilde{x}^{\text{set}}(t)\big)^\top\big(\tilde{x}^{\text{set}}(t)\big)}_{\leq\text{trace}\left(X(t)\right)}\\ &\leq\text{trace}\big(P(t)+X(t)\big)\ .\end{aligned}\tag{8}$$

Thus, the MSE is bounded by $\text{trace}\big(P(t)+X(t)\big)$. The estimator proposed in [10] forms the basis of the EBSE-NI, as it minimizes exactly this bound. Contrary to standard Kalman filtering, the estimate $\hat{x}(t)$ is not only associated to an error-covariance $P(t)$ but also to a shape matrix $X(t)$.

A. Filtering

As it is done in standard Kalman filtering, the EBSE-NI can be initialized with a prior estimate $\hat{x}(0)$ and error-covariance $P(0)$, while $X(0)$ can be set to zero. At an instant t, let $\hat{x}(t^-)$ denote the current (prior) estimate with error matrices $P(t^-)$ and $X(t^-)$ that is to be combined with a measurement $z(t)$ related to the model (7). A linear estimator

$$\hat{x}(t)=K_1\hat{x}(t^-)+K_2z(t)\tag{9}$$

is desired that minimizes the MSE bound in (8). The estimator must not be biased since a bias $\eta(t)$ would increase (8) by the positive value $\|\eta(t)\|_2$. Consequently, this implies $K_1=I-KC$ with $K=K_2$. For a yet-to-be-determined gain K, the posterior error-covariance yields

$$P(t)=(I-KC)P(t^-)(I-KC)^\top+KVK^\top\,,\tag{10}$$

and the posterior ellipsoid matrix is given by

$$X(t)=\tfrac{1}{1-\omega}(I-KC)X(t^-)(I-KC)^\top+\tfrac{1}{\omega}KEK^\top\,.\tag{11}$$

The parameter $\omega\in(0,1)$ in (11) guarantees that the shape matrix $X(t)$ corresponds to an outer ellipsoidal approximation of two ellipsoidal sets each characterized by the shape matrices $(I-KC)X(t^-)(I-KC)^\top$ and $KEK^\top$, respectively. According to [10], it can be shown that the Kalman gain is given by

$$\begin{aligned}K=&\Big(P(t^-)C^\top+\tfrac{1}{1-\omega}X(t^-)C^\top\Big)\cdot\\ &\Big(CP(t^-)C^\top+\tfrac{1}{1-\omega}CX(t^-)C^\top+V+\tfrac{1}{\omega}E\Big)^{-1}\,.\end{aligned}\tag{12}$$

Hence, a one-dimensional convex optimization problem for $\omega^{\text{opt}}\in(0,1)$ remains to be solved that minimizes the posterior MSE bound in (8), e.g., with the aid of Brent's method.

Remark V.1 The derived gain (12) embodies a systematic and consistent generalization of the standard Kalman filter for additional unknown but bounded uncertainties. Accordingly, K in (12) reduces to the standard Kalman gain in the absence of set-membership errors, i.e., $X(t^-)=0$ and $E=0$.

Remark V.2 The employed estimator is closely linked to split covariance intersection [16] when $X(t)$ and E are considered to be covariance matrices of stochastic errors with unknown correlations. Furthermore, the sum of the error matrices can be rewritten to

$$\begin{aligned}P(t)+X(t)=&\Big(\omega\big(\omega P(t^-)+X(t^-)\big)^{-1}\\ &+(1-\omega)\big((1-\omega)V+E\big)^{-1}\Big)^{-1}\,,\end{aligned}$$

which can be utilized to determine ω^{opt} that minimizes the right-hand side bound in (8). Hence, the special cases $\omega^{\text{opt}}=0$ or $\omega^{\text{opt}}=1$ only occur if $X(t^-)$ or E are zero matrices.

As indicated earlier, the filtering step of the EBSE-NI depends on the two different types of measurement information at the two different types of sampling instants:

1) Event triggered instant: When an event is triggered at $t=t_{\mathsf{e}}$, i.e., $t\in\mathbb{T}_e$, the current measurement is instantaneously disclosed to the estimator. In this case, no set-membership uncertainty $e(t)$ is present, i.e., $\mathbb{L}_{0,E(t)}=\emptyset$ and thus $E=0$ and $z(t)=y(t_{\mathsf{e}})$. Hence, (7) coincides with the original model (1b). The gain K in (12) can be simplified to

$$\begin{aligned}K=&\Big(P(t^-)C^\top+X(t^-)C^\top\Big)\cdot\\ &\Big(CP(t^-)C^\top+CX(t^-)C^\top+V\Big)^{-1},\quad\forall t\in\mathbb{T}_e\end{aligned}$$

which corresponds to the parameter value $\omega=0$. Therefore, no numerical optimization is required. In particular, the posterior shape matrix $X(t)$ in (11) becomes

$$X(t)=(I-KC)X(t^-)(I-KC)^\top,\quad\forall t\in\mathbb{T}_e\,.$$

It can be expected that the value of this shape matrix is significantly decreased at the event instants, since no set-membership uncertainty is ascribed to $z(t)$.

2) Periodic time instant: Set-membership uncertainty comes into play whenever no measurement is received by the estimator. Proposition III.1 states that the actual measurement lies in a set $\mathbb{H}(\mathsf{e},t)$, which is the only measurement information available, i.e., $y(t)=Cx(t)+v(t)\in\mathbb{H}(\mathsf{e},t)$. A characterization of $\mathbb{H}(\mathsf{e},t)$ for the discussed Send-on-Delta and Matched Sampling strategies has been derived in (3) and (5), respectively. Now, instead of the actual but unknown measurement $y(t)$, the estimator relies on the implied measurement $z(t)=y(t)+e(t)$. Therein, $e(t)$ has been introduced so to model the set-membership property $y(t)\in\mathbb{H}(\mathsf{e},t)$, due to which a realization of $e(t)\in\mathbb{L}_{0,E(t)}$ will depend on $\mathbb{H}(\mathsf{e},t)$. Note that apart from attributing a set-membership error to $z(t)$, the stochastic noise in model (7) has to be taken into consideration as well as the event triggering criteria is based on the noise sensor reading $y(t)$ rather than $Cx(t)$. Consequently, the estimator must take both $v(t)$ and $e(t)$ into account. Let us illustrate the implied measurement $z(t)$ for the considered sampling strategies, next

- *Send-on-Delta*: The true measurement lies in the neighborhood of $y(t_{\mathsf{e}-1})$, i.e., $\|y(t) - y(t_{\mathsf{e}-1})\|_2 \leq \Delta$. More precisely, $y(t)$ lies in a ball of radius Δ around $y(t_{\mathsf{e}-1})$ and can thus be characterized by an ellipsoidal set with an ellipsoidal shape matrix $E = \Delta^2 I$. Hence, the estimator can conduct a filtering step with the last received measurement $z(t) = y(t_{\mathsf{e}-1})$ when the corresponding measurement error $z(t) - y(t_{\mathsf{e}-1}) = v(t) + e(t)$ is composed of a stochastic part $v(t)$ and set-membership part $e(t) \in \mathbb{L}_{0,E}$.
- *Matched Sampling*: Again, the last event-triggered information at $t_{\mathsf{e}-1}$ is utilized. The estimate $\hat{x}(t_{\mathsf{e}-1})$, which was updated with the last event-triggered measurement, is predicted to current time t according to $\hat{x}(t^-) = A_{\tau_{\mathsf{e}}}\hat{x}(t_{\mathsf{e}-1})$. The time-periodic filtering step is then carried out with $z(t) = C\hat{x}(t^-)$. The measurement error $z(t) - C\hat{x}(t^-) = v(t) + e(t)$ now comprises the additional set-membership component $e(t) \in \mathbb{L}_{0,E}$, where $E = \Phi$ is given by (5).

For both discussed sampling strategies, the time-periodic measurements are associated to a stochastic and a set-membership error term. The gain K in (12) can then be determined in order to compute the posterior mean $\hat{x}(t)$ in (9), error-covariance $P(t)$ in (10), and error ellipsoid matrix $X(t)$ in (11).

B. Prediction

The Kalman prediction step is carried out by means of the process model (1a). An estimate $\hat{x}(t-\tau)$ at the periodic sampling instant $t-\tau$ is transformed to a prior estimate $\hat{x}(t^-)$ at instant t according to

$$\hat{x}(t^-) = A_\tau \hat{x}(t-\tau)\,.$$

The zero-mean process noise $w(t-\tau)$ only affects the predicted error-covariance, which yields

$$P(t^-) = A_\tau P(t-\tau) A_\tau^\top + B_\tau W B_\tau^\top\,.$$

In contrast to a standard Kalman filter, also the corresponding error ellipsoid matrix has to be updated, i.e.,

$$X(t^-) = A_\tau X(t-\tau) A_\tau^\top\,. \tag{13}$$

Evidently, the predicted parameters can be computed in closed form and the prediction step only differs from the standard Kalman filter by the additional equation (13).

VI. Illustrative case-study

The effectiveness of the developed EBSE-NI is illustrated in terms of its estimation error in a 1D object tracking example. The process model in line with (1) is a double integrator, i.e.,

$$\begin{aligned} x(t) &= \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} \frac{1}{2}\tau^2 \\ \tau \end{bmatrix} a(t-\tau)\ , \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v(t)\ . \end{aligned}$$

The state vector $x(t)$ combines the object's position and speed. Further, $a(t) = \frac{1}{30}t \cdot \cos(\frac{1}{10}t)$ denotes the object's acceleration, while only the position is measured in $y(t)$. Since acceleration is assumed unknown, the process model in (1) is characterized with a process noise $w(t) := a(t)$. As $|a(t)| \leq 0.9$, for the simulated period $t < 25$ seconds, a suitable covariance in line with [17] is $cov(a(t)) = 1.1$ resulting in an unbiased distribution $p(w(t))$ with covariance $W = 1.1$. Further, the sampling time is $\tau_s = 0.1$ seconds and the sensor noise covariance is set to $V = 2 \cdot 10^{-3}$. The object's true position, speed, and acceleration are depicted in Figure 6.

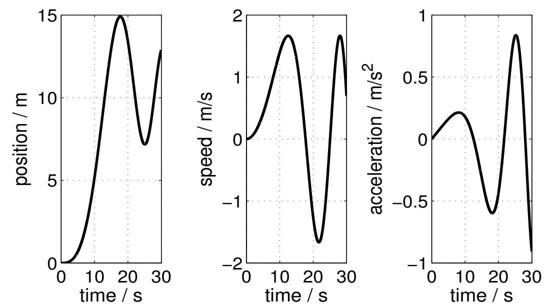

Fig. 6. The position, speed, and acceleration of the tracked object.

The proposed event-based state-estimator with negative information (EBSE-NI) combining stochastic and set-membership measurement information is compared to an existing event-based state estimator (EBSE) presented in [6] limited to stochastic measurements. Both estimators start with the initial estimation results $\hat{x}(0) = \begin{pmatrix} 0.1 & 0.1 \end{pmatrix}^\top$ and $P(0) = 0.01 \cdot I$, while $\mathbb{X}(0) = 0$ is chosen as the initial ellipsoidal shape matrix for the proposed EBSE-NI. Next, the measurement information of both EBSEs is characterized.

EBSE-NI

Measurement information of the proposed EBSE-NI is represented by the implied measurement $z(t) = y(t) + e(t)$, where $y(t)$ is the actual measurement value and $e(t) \in \mathbb{L}_{0,E}$ is bounded by the ellipsoidal set $\mathbb{L}_{0,E(t)}$ characterized by the shape matrix $E(t) \in \mathbb{R}_+$ (a scalar value). In case of an event instant $t \in \mathbb{T}_e$ the measurement $y(t_{\mathsf{e}})$ is received and one obtains that $z(t) = y(t_{\mathsf{e}})$, i.e., $e(t_{\mathsf{e}}) \in \emptyset$ and $E(t) = 0$. At periodic time instants $t \in \mathbb{T}_p$ one has the information that $y(t) \in \mathbb{H}(\mathsf{e}, t)$, where $\mathbb{H}(\mathsf{e}, t)$ depends on the employed sampling strategy. This ellipsoidal set $\mathbb{H}(\mathsf{e}, t)$ can be characterized with a "mass"-center, yielding an estimate of $y(t)$, and an ellipsoidal error-set resulting in an characterization of $e(t) \in \mathbb{L}_{0,E(t)}$ and thus of the shape matrix $E(t)$. A suitable characterization of $y(t)$ and $E(t)$ for the two employed event sampling strategies presented in Section III given $\Phi(t)$ as introduced in (5), yields

$$\begin{aligned} \text{SoD: } & z(t) = y(t_{\mathsf{e}})\,, & E(t) &= 0\,, & \forall t &\in \mathbb{T}_e, \\ & z(t) = y(t_{\mathsf{e}-1})\,, & E(t) &= \Delta^2\,, & \forall t &\in \mathbb{T}_p, \\ \text{MS: } & z(t) = y(t_{\mathsf{e}})\,, & E(t) &= 0\,, & \forall t &\in \mathbb{T}_e, \\ & z(t) = CA_{t-t_{\mathsf{e}-1}}\hat{x}(t_{\mathsf{e}-1})\,, & E(t) &= \Phi(t)\,, & \forall t &\in \mathbb{T}_p. \end{aligned}$$

EBSE

Measurement information of the EBSE presented in [6] is represented by a Gaussian PDF, i.e., $p(y(t)) = \mathcal{G}\big(y(t), \hat{y}(t), R(t)\big)$ for some (estimated) measurement value $\hat{y}(t)$ and covariance matrix $R(t)$. Then, a standard (aperiodic) Kalman filtering routine can be performed for updating the estimation results by considering $\hat{y}(t)$ as the measurement value and $V + R(t)$ as the sensor noise covariance matrix. In case of an event instant $t \in \mathbb{T}_e$ the measurement $y(t_{\mathsf{e}})$ is received and one obtains that $\hat{y}(t) = y(t_{\mathsf{e}})$ and $R = 0$. At periodic time instants $t \in \mathbb{T}_p$ one has the information that $y(t) \in \mathbb{H}(\mathsf{e}, t)$, which is then turned into a particular value for $\hat{y}(t)$ and $R(t)$. A suitable characterization of $\hat{y}(t)$ and $R(t)$ for the two employed event sampling strategies presented in Section III given $\Phi(t)$ as introduced in (5), yields

$$
\begin{aligned}
\text{SoD: } & \hat{y}(t) = y(t_{\mathsf{e}})\,, && R(t) = 0\,, && \forall t \in \mathbb{T}_e,\\
& \hat{y}(t) = y(t_{\mathsf{e}-1})\,, && R(t) = \frac{3}{4}\Delta^2\,, && \forall t \in \mathbb{T}_p,\\
\text{MS: } & \hat{y}(t) = y(t_{\mathsf{e}})\,, && R(t) = 0\,, && \forall t \in \mathbb{T}_e,\\
& \hat{y}(t) = CA_{t-t_{\mathsf{e}-1}}\hat{x}(t_{\mathsf{e}-1})\ , && R(t) = \frac{1}{4}\Phi(t)\,, && \forall t \in \mathbb{T}_p.
\end{aligned}
$$

Figure 7 until Figure 10 depict the actual squared estimation error, i.e., $\|\hat{x}(t) - x(t)\|_2^2$, in comparison to the modeled estimation error, i.e., $\mathrm{tr}(P(t))$ for the alternative EBSE and $\mathrm{tr}(P(t)) + \mathrm{tr}(X(t))$ for the proposed EBSE-NI. The results depicted were obtained after averaging the outcome of 1000 runs of the considered simulation case study.

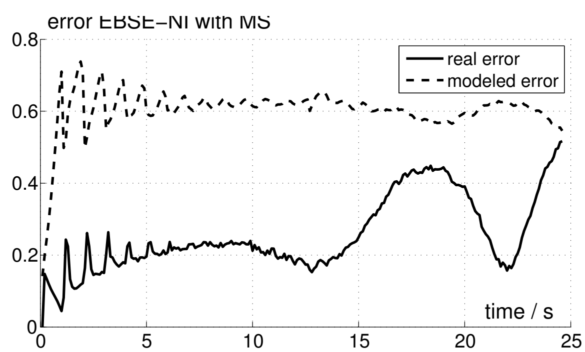

Fig. 7. Simulation results for *Matched Sampling* in combination with the *proposed EBSE-NI* exploiting stochastic and set-membership information. The real squared estimation error $\|\hat{x}(t) - x(t)\|_2^2$ is depicted versus the modeled (bound) of the estimation error $\mathrm{tr}(P(t)) + \mathrm{tr}(X(t))$.

Figure 7 and Figure 8 depict the estimation results of the proposed EBSE-NI and the alternative EBSE, respectively, when Matched Sampling is employed as the event sampling strategy. Although it is not pointed out in the figures, it is worth mentioning that the proposed EBSE-NI triggered a total amount of 31 events (on average), while the alternative EBSE triggered 40 events (on average). Still, the real squared estimation error of both EBSEs considered is comparable. Hence, the proposed EBSE has similar estimation results with fewer events triggered, due to which less measurement samples are required saving communication resources. Yet, the main advantage of the proposed EBSE-NI is the modeled

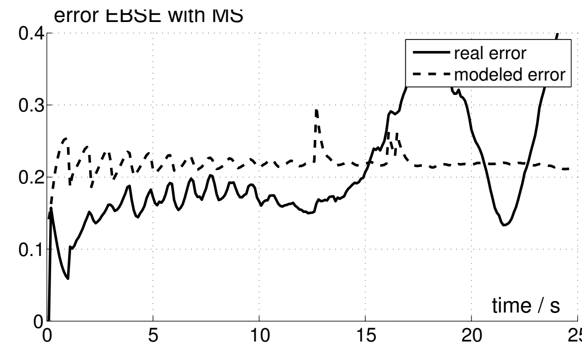

Fig. 8. Simulation results for *Matched Sampling* in combination with the *alternative EBSE* exploiting stochastic information, only. The real squared estimation error $\|\hat{x}(t) - x(t)\|_2^2$ is depicted versus the modeled (bound) of the estimation error $\mathrm{tr}(P(t))$.

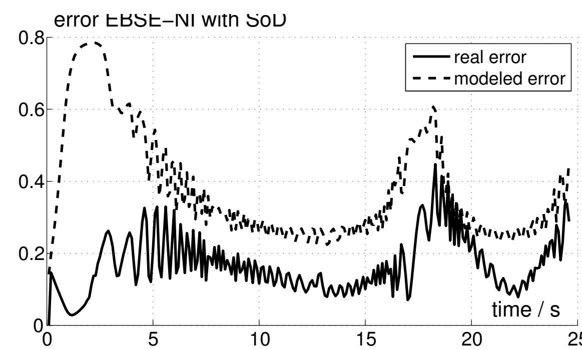

Fig. 9. Simulation results for *Send-on-Delta* in combination with the *proposed EBSE-NI* exploiting stochastic and set-membership information. The real squared estimation error $\|\hat{x}(t) - x(t)\|_2^2$ is depicted versus the modeled (bound) of the estimation error $\mathrm{tr}(P(t)) + \mathrm{tr}(X(t))$.

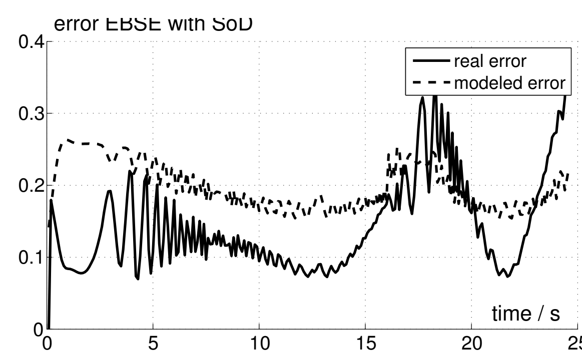

Fig. 10. Simulation results for *Send-on-Delta* in combination with the *alternative EBSE* exploiting stochastic information, only. The real squared estimation error $\|\hat{x}(t) - x(t)\|_2^2$ is depicted versus the modeled (bound) of the estimation error $\mathrm{tr}(P(t))$.

bound on the estimation error. Figure 7 indicates that this modeled bound is conservative when the proposed EBSE-NI is employed, which is not the case for the alternative EBSE depicted in Figure 8. This means that the EBSE-NI gives a better guarantee that the real estimation error stays within the bound as it is computed by the estimator. Such a property is important when estimation results are used for control purposes, since control stability of a networked system relies on such a property.

Figure 9 and Figure 10 depict the estimation results of the proposed EBSE-NI and the alternative EBSE, respectively, when Send-on-Delta is employed as the event sampling strategy. Since this event sampling strategy does not depend on previous estimation results but merely on the previous measurement sample, the events for both estimators were triggered at the same time instants giving a total of 115 events. Note that this is an increase of events by a factor of 3 to 4 when compared to the EBSEs in combination with Matched Sampling. Yet, this increase of events and thus of measurement samples is not reflected in a corresponding decrease of estimation errors. Further, similar conclusions can be drawn from the estimation results with Send-on-Delta when comparing Figure 9 and Figure 10. Again, the squared estimation error of the two considered EBSEs is comparable and the main advantage of the proposed EBSE-NI is in the modeled bound yielding a better guarantee on the real estimation error.

Therefore, a fair conclusion of the proposed EBSE-NI is that similar estimation errors are achieved when compared to a state-of-the-art alternative estimator, although the *modeled* estimation error of the proposed EBSE-NI is a far better bound on *real* estimation errors. As such, the proposed EBSE is advantageous in networked control systems where estimation results are being used by a (stabilizing) controller.

VII. Conclusions

In networked systems, high measurement frequencies may rapidly exhaust communication bandwidth and power resources when sensor data must be transmitted periodically to the state estimator. The transmission rate can significantly be reduced if an event-based strategy is employed for sampling sensor data. "Send-on-Delta" and "Matched Sampling" have been discussed as examples of such strategies. The estimation system can perform a measurement update whenever an event is triggered, i.e., a new measurement is received. However, estimation algorithms are, in general, intended to compute and provide estimates periodically. Of course, the time gap between events can simply be bridged by prediction steps, but additional knowledge then remains untapped: *as long as no event is triggered, the actual measurement does not fulfill the event-sampling criterion, which has been called negative information.* For the considered sampling strategies, the corresponding criteria can directly be translated into set-membership information. Although sensors are commonly modeled to be purely affected by a stochastic error, the negative information at periodic time steps is related to a second set-membership uncertainty. With a recently proposed combined stochastic and set-membership Kalman filter, both types of uncertainties can be incorporated. In simulations, this estimator provides reliable error bounds and has been compared to a purely stochastic approach, where the set-membership has been represented by a probability density and the error has then been underestimated.

Prospective research focuses also on unreliable networks, where delays and packet losses have to be taken into account. Delayed state filtering techniques may then have to be combined with event-based strategies. It is also possible to introduce a weighting parameter between the stochastic and set-membership errors. This parameter may contribute to reducing the overall estimation uncertainty. So far, only stochastic systems and sensors have been considered. With the combined stochastic and set-membership Kalman filter, also unknown but bounded errors affecting sensor readings and control inputs, such as discretization and linearization errors, can be treated.

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