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IMPROVED "BOCR" ANALYSIS WITH THE AHP/ANP

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Summary: This paper shows that the usual multiplicative synthesis of alternative priorities for benefits, opportunities, costs and risks, obtained from separate Analytic Hierarchy or Network models, can be ambiguous. The ratio of benefit and opportunity priorities to cost and risk priorities can be misleading when assessing the profitability of a project. The same holds for their additive synthesis, although advocated by AHP/ANP. A quotient of these priorities with weights as coefficients, not powers, will however produce sound results, provided that the four separate models are properly related to each other by weights that make the priorities on the four factors commensurate and are obtained from magnitude comparisons.

1. Introduction

A decision on whether or not to undertake a project usually requires investigating the positives (benefits) and negatives (costs) of that project and an attempt to express those in monetary terms such as dollars. If that project has a benefit/cost ratio > 1, its benefits outweigh its costs. If there are several projects to choose among, those projects will usually be ordered according to their respective benefit/cost ratios. The ones with a ratio > 1 are the attractive ones exceeding the break-even point, and the one with the highest ratio among these gives the highest return on money spent and would very likely be the one to be chosen.

The problem is that often benefits and costs are difficult to express in monetary terms, especially when some of the benefits or costs are intangible, such as "improved accuracy" or "learning efforts". The Analytic Hierarchy Process (AHP) developed by Saaty (1980) has been advocated as an approach that not only can deal with both tangibles and intangibles but also helps organize all aspects involved in a hierarchic structure where the benefit or cost aspects act as criteria and the projects as alternatives. Usually, we have separate hierarchies: one costs hierarchy and one benefits hierarchy. One has to pair-wise compare the importance of cost criteria in the cost hierarchy, and the same with respect to the benefit criteria in the separate benefits hierarchy. These processes produce relative criteria weights expressed on a derived ratio scale, usually normalized to the unity sum for each family of criteria in each hierarchy. The alternative projects are pairwise compared with respect to each criterion on the lowest level of each hierarchy; their derived priorities are expressed on a ratio scale as well, again usually normalized to the unity sum per criterion. Synthesis of the alternative priorities and the criteria weights using a weighted sum produces composite alternative priorities for each hierarchy. For each alternative, its composite benefit priority is then divided by its composite cost priority. The resulting ratio value serves as a means to rank the alternatives and choose the best one, i.e. the alternative with the highest benefit/cost-priority ratio. Examples of benefit/cost analysis using the AHP have been published in Saaty (1980, 1994).

Benefit/cost analysis with the AHP has been criticized though. The main criticism is that different hierarchies produce priorities on different derived ratio scales which are usually not commensurate. The quotient of two ratio scales is again a ratio scale but has lost its clear relationship with the individual scales. The result may therefore appear meaningful as a measure of profitability, whereas it is in fact not: benefit/cost-priority ratios may be larger than unity when in fact the costs exceed the benefits.

Wedley et al (2003) have reviewed previous literature where the AHP was adapted for proper benefit/cost analysis, including suggestions by Saaty (1994; page 151) to produce more meaningful ratios. Wedley et al. (2001) suggested a formal magnitude adjustment procedure that converts the benefit and cost hierarchies to a common unit thus assuring that resulting benefit/cost ratios do have the desirable property of correctly indicating the break-even point. The questioning procedures they proposed are however cognitively difficult. Wedley et al. (2003) proposed using linking pin methods, thereby potentially easing the questions for relating benefits to costs.

In the AHP literature thus far, only risks (R) have been added to the B/C ratio, see for example Saaty (1994, pp. 164-166): for each alternative a B/(C*R) ratio is computed based on priorities that are obtained from three different hierarchies. In theory, a fourth factor, opportunities (O), can be added to the analysis, thus allowing a full BOCR analysis where positives not only include benefits but opportunities as well, and negatives not only costs but also risks.

Recently, the Analytic Network Process with supporting software was developed by Saaty (2001), enabling one to model systems with feedback and dependence. Especially in the context of the ANP, many applications of BOCR analysis are offered where a network model for each of the four BOCR factors is set up instead of a hierarchy. This allows for modeling interrelationships between the elements defining each of the four overall factors.

Not only B/C analysis but also B/(C*R) analysis has been criticized by, for example, Millet & Wedley (2003). These authors argue that the product of costs and risks is not meaningful or justified, with the added argument that differences in relative importance are not accounted for. Similar arguments may hold for BOCR analysis. In this paper however, BOCR analysis will not be questioned per se; it is rather the computational method for getting results that is the paper’s subject.

This paper investigates the way in which the priorities of alternatives on each of the four factors (B, O, C, R) are synthesized. It is shown, supported by a validation example using monetary values, that one cannot be certain that the multiplicative or additive expressions for synthesis proposed by Saaty (2001) produce the correct ordering of alternatives, or give a reliable indication of profitability, except in special cases. This is due to incommensurability of the composite priorities on the four factors. In the context of B/(C*R) analysis, Millet & Wedley (2003) have touched upon this issue without elaborating it.

This paper further argues that (a) for synthesis the composite priorities of each of the four factors should be expressed on a commensurate scale by using rescaling weights, and (b) a revised synthesis formula should be used that is not additive but a quotient of positives to negatives with the rescaling weights as coefficients, not powers, and, finally, (c) if weights based on personal values are to be used, they should be applied to the rescaled priorities. The weights that rescale the composite priorities are based on magnitude comparisons between the four factors.

The remainder of this paper is organized as follows. First, the reference example is introduced, with some additional assumptions. Then the results using the formulas for synthesis proposed in Saaty (2001) are shown and critically discussed, after which the improved approach is presented in several steps. The paper concludes with a summary of the merits of the improved approach, and attempts to draw a generally valid conclusion.

2. Reference example of BOCR analysis

In order to investigate the methods for synthesizing the composite priorities on each of the four BOCR factors, an arbitrary example was taken from Saaty (2001, section 5-7) involving the development of a condominium.

The analysis is conducted as follows, in Saaty’s words (Saaty, 2001, pages 182-183):

> “… First, we develop priorities for personal values. Next, we rate each of the four BOCR merits on the personal values. Third, we create and prioritize the control criteria for each of the BOCR, and finally, we create and prioritize the decision networks for each of these control criteria. To obtain the answer we synthesize the priorities of the alternatives for benefits and then for opportunities and then for costs and then for risks, thus obtaining four different rankings for each alternative. We use the priorities of BOCR to weight and synthesize the overall weights of the alternatives obtained from the four merit structures. In this

process we must use [for additive synthesis][1] the reciprocals of the synthesized final priorities of the alternatives under costs and risks obtaining high priorities for the least costly, the most risky …"

Instead of simply using the BO/CR ratio of multiplied priorities to evaluate the alternatives in the final decision, Saaty suggests two expressions for synthesizing the composite priorities with the use of weights that allow accounting for differences in relative importance of the factors from a "personal" view.

Multiplicative with weights as powers:

$$\frac{B_p^{w_b} * O_p^{w_o}}{C_p^{w_c} * R_p^{w_r}} \tag{1}$$

Additive with weights as coefficients:

$$w_b * B_p + w_o * O_p + w_c * {}^{1}\!/_{C_p^*} + w_r * {}^{1}\!/_{R_p^*} \tag{2}$$

where:
B_p, O_p, C_p and R_p are the normalized overall priorities of the alternatives on benefits, opportunities, costs and risks respectively, computed using underling hierarchic or network models; and

${}^{1}\!/_{C_p^*}$, ${}^{1}\!/_{R_p^*}$

are the normalized reciprocals of C_p and R_p; and w_b, w_o, w_c and w_r are the normalized weights for each of the four factors, respectively, established using personal values arguments.
Expression (2) is an approximation of (1) (page 188 in Saaty (2001)), valid for values of (1) lying between zero and two, and for weights w_k ($k \in \{b,o,c,r\}$) normalized to the unity sum, although Saaty cautions that the two outcomes may not always be close. More generally, Saaty & Hu (1998) have shown that multiplicative synthesis $\prod_i x_i^{w_i}$ and additive synthesis $\sum_i x_i^{w_i}$ may be related analytically through approximation but the outcomes can be different and may even lead to different rankings.
Note further that both nominator and denominator in (1) use a product and not a sum of factors. This will be addressed again in section 6.

The computations which produce the final priorities on each of the BOCR factors are of no concern here. The weights of the BOCR factors proper, the normalized composite priorities of the alternatives on each of the factors, and the normalized inverse priorities on costs and risks, are shown in Table 1. This table is based on Tables 5-26 and 5-27 in Saaty (2001).

Table 1: BOCR weights (established using personal values) and alternative priorities (from underlying models)

Composite priorities on factors	Benefits (0.184) B_p	Opportunities (0.263) O_p	Costs (0.228) C_p	Risks (0.326) R_p	1/Costs (0.228) ${}^{1}\!/_{C_p^*}$	1/Risks (0.326) ${}^{1}\!/_{R_p^*}$
Alt. A1	0.097	0.112	0.191	0.167	0.514	0.552
Alt. A2	0.461	0.356	0.391	0.374	0.251	0.247
Alt. A3	0.442	0.532	0.418	0.459	0.235	0.201
sum	1.000	1.000	1.000	1.000	1.000	1.000

The overall composite alternative priorities computed by either expression are shown in Table 2.
It can be observed from Table 2 that the rank orders of the alternatives differ when doing multiplicative and additive synthesis:

A3 > A2 > A1 (multiplicative),
A1 > A3 > A2 (additive).

There is no point in arguing which ordering is correct, as we do not know underlying objective values and the weights are merely reflections of one's subjective feelings of the relative importance of the BOCR merits.
There may be a point, however, in arguing which type of synthesis, multiplicative or additive, is more

[1] Insertion by this paper's author.

justified. Saaty (2001) favors the additive expression over the multiplicative one as it is more in conformity with hierarchic and network composition, supported by an example of buying a house where multiplicative synthesis does not reproduce the correct results with known monetary values. The next section will discuss the merits of both.

Table 2: Final (Overall) composite priorities using two synthesis expressions and "personal" BOCR weights (ref. Table 5-28 in Saaty (2001))

Final composite priorities	Weighted multiplicative	Normalized weighted multiplicative	(Normalized) weighted additive
Alt. A1	0.957	0.296	**0.344**
Alt. A2	1.128	0.349	0.316
Alt. A3	1.146	**0.355**	0.340
sum	3.231	1.000	1.000

In Wedley et al. (2001 and 2003) it is shown with regard to B/C analysis that results using priorities from different hierarchies can be deceiving in terms of profitability of the projects. In order to investigate whether the same is true with BOCR analysis, actual dollar figures are ascribed to benefits, costs, opportunities and risks for each alternative, but in such a way that the original priorities of Table 1 are reproduced. In reality, establishing monetary values for the factors may be a heavy burden due to intangibles and perhaps complicated dependencies involved, which is the prime reason for using AHP/ANP. It is therefore only for reasons of validation that dollar figures are assigned to the four factors. Using examples with known outcomes has appeared to be well accepted, both in single criterion and multiple criteria situations; Saaty (1994, chapter 9; 2001, section 5-2), Vargas (1997), R. Saaty (2004), among others (including the "Wedley Singapore study"), use examples with known values for validation purposes or for denouncing multiplicative synthesis.

Table 3: Monetary scenarios representing different situations, but all producing the priorities of Table 1

	Alt. A1	Alt. A2	Alt. A3	Total
Benefits in \$ (B_m) Opportunities in \$ (O_m)	388 224	1844 712	1768 1064	4000 2000
Scenario 1: Costs in \$ (C_m) Risks in \$ (R_m) Ratio $(B_m{*}O_m)/(C_m{*}R_m)$ Total multiplied negatives exceed total multiplied positives ("BO<CR"); unprofitable alternatives	1528 334 0.170	3128 748 0.561	3344 918 0.613	8000 2000 0.5
Scenario 2: Costs in \$ (C_m) Risks in \$ (R_m) Ratio $(B_m{*}O_m)/(C_m{*}R_m)$ Total multiplied negatives equal total multiplied positives ("BO=CR"); mixed profitability of alternatives	1528 167 0.341	3128 374 1.122	3344 459 1.226	8000 1000 1.0
Scenario 3: Costs in \$ (C_m) Risks in \$ (R_m) Ratio $(B_m{*}O_m)/(C_m{*}R_m)$ Total multiplied positives exceed total multiplied negatives ("BO>CR"); profitable alternatives	1146 16.7 4.541	2346 37.4 14.964	2508 45.9 16.341	6000 100 13.333

In Table 3 three monetary scenarios are shown. They are characterized by the same benefit and opportunity amounts but different amounts for costs and risks. All scenarios produce the priorities shown in Table 1; on each of the four factors, the normalized relative priorities of the alternatives remain unchanged. Scenarios 1, 2 and 3 model three different situations regarding the products of total benefit and opportunity amounts on the one hand and of total cost and risk amounts on the other hand. Moreover, in the first scenario all individual ratios are less than 1 suggesting that all alternatives are unprofitable, in the third scenario all ratios are greater than 1, suggesting that all alternatives are indeed profitable, while the second scenario represents a mixed situation for the alternatives.

3. Comparison of dollar-based and priority-based results

Let us take a closer look at scenarios 1, 2 and 3 in Table 3. The results suggest that in all scenarios the rank order of the alternatives is: A3 > A2 > A1. Profitability of the alternatives appears to be very different from one scenario to another. In scenario 1, A3 may be the best alternative but it is certainly not a profitable one as its monetary BOCR ratio is (far) below the break-even point. In that case, choosing A3 would be a bad decision, even if we would *add* monetary B & O values and C & R values. In fact, from an additive perspective all projects are deficient in all scenarios except A2 and A3 in scenario 3.

In order to fairly compare the monetary results of Table 3 with those based on priorities, Table 4 shows unweighted priority-based results, or rather equally weighted priorities, instead of the "personally" weighted ones in Table 2. In order to comply with the conditions producing the approximation in (2) and thus in order to fairly compare the results of the two expressions, the weights of the four BOCR factors have been put equal to 0.25. Using equal weights instead of no weights at all does of course not affect the rank order of the alternatives or the indication of profitability in multiplicative synthesis.

$$\{(B_p * O_p)/(C_p * R_p)\}^{0.25} \qquad \text{(multiplicative)} \qquad (3)$$

$$0{,}25 * (B_p + O_p + 1/C_p^* + 1/R_p^*) \qquad \text{(additive)} \qquad (4)$$

Table 4: Overall composite priorities using two synthesis expressions and equal BOCR weights

	Alt. A1		Alt. A2		Alt. A3	
	outcome	normalized	outcome	normalized	outcome	normalized
multiplicative	0.764	0.268	1.029	0.362	1.052	0.370
additive	0.319	0.319	0.329	0.329	0.352	0.352

The outcomes in Table 4 using the *multiplicative* synthesis expression (3) suggest that both A2 and A3 are profitable alternatives (ratio larger than 1), whereas in monetary scenario 1 they would not, and A1 is unprofitable (ratio less than 1), whereas in scenario 3 it would be a profitable alternative. Only if scenario 2 were to represent the "true" values, would the priority-based multiplicative results provide an adequate profitability indication.

There is no reason why the equally weighted *additive* expression (4) should produce a correct indication of profitability other than by chance, since its outcomes obviously always sum to 1 over the alternatives when each of the factors is normalized to the unity sum. This is also true for any other weighting scheme with the additive expression.

Thus neither expression is a reliable indicator of profitability with the only exception of equally weighted multiplicative synthesis in the case of the nominator and the denominator being equal (ref. scenario 2).

It will be shown hereafter that weighted multiplicative synthesis of BOCR priorities, using any weighting scheme $\{v_b, v_o, v_c, v_r\}$ reproduces the equivalently weighted monetary ratios, if and only if the nominator and denominator of that weighted monetary ratio of totals are equal; i.e. $(B_m^t)^{v_b} * (O_m^t)^{v_o} = (C_m^t)^{v_c} * (R_m^t)^{v_r}$.

$$\frac{(B_p^i)^{v_b} * (O_p^i)^{v_o}}{(C_p^i)^{v_c} * (R_p^i)^{v_r}} = \frac{(B_p^i)^{v_b} * (O_p^i)^{v_o}}{(C_p^i)^{v_c} * (R_p^i)^{v_r}} * \frac{(B_m^t)^{v_b} * (O_m^t)^{v_o}}{(C_m^t)^{v_c} * (R_m^t)^{v_r}} = \frac{(B_p^i * B_m^t)^{v_b} * (O_p^i * O_m^t)^{v_o}}{(C_p^i * C_m^t)^{v_c} * (R_p^i * R_m^t)^{v_r}} = \frac{(B_m^i)^{v_b} * (O_m^i)^{v_o}}{(C_m^i)^{v_c} * (R_m^i)^{v_r}} \qquad (5)$$

where the superscript *t* of a quantity refers to the total of the alternatives and the superscript *i* refers to any individual alternative. Perfect consistency is assumed when prioritizing using pairwise comparisons.

Furthermore, the weighted multiplicative expression for priority synthesis always produces the correct ordering, with any set of weights. The proof of this is straightforward and similar to (5).

Only the equally weighted priority-based ordering also correctly reflects the ordering based on un-weighted monetary values, due to the continuously increasing nature of the power function. Weighted multiplicative synthesis of priorities may well reproduce its monetary equivalent, but only in the case of equal weights is that quotient a meaningful indicator of profitability based on break-even analysis using the unity value of that quotient.

It can further be observed from Table 4 that using expressions with equal weights has produced

equivalent rank orderings from both synthesis modes; the best alternative can now be identified without ambiguity in this example. They are identical to the rank ordering obtained from each monetary scenario.
The equally weighted additive expression (4) does not always produce the correct ordering, as a new example in Table 5 shows: the correct ordering is A1 > A2 > A3, whereas (4) yields A2 > A3 > A1.

Table 5: Example showing that the approximate additive synthesis expression does not (always) provide the correct ordering of alternatives

	Alt. A1	Alt. A2	Alt. A3
Benefits in \$ (B_m) Benefit priorities (B_p)	2000 0.2	7500 0.75	500 0.05
Opportunities in \$ (O_m) Opportunity priorities (O_p)	3000 0.3	6500 0.65	500 0.05
Costs in \$ (C_m) Cost priorities (C_p)	5000 0.5	4500 0.45	500 0.05
Risks in \$ (R_m) Risk priorities (R_p)	500 0.05	9000 0.9	500 0.05
Unweighted monetary ratio $(B_m*O_m)/(C_m*R_m)$ Equally weighted monetary ratio $(B_m*O_m)/(C_m*R_m)^{0.25}$ Equally weighted priority ratio $(B_p*O_p)/(C_p*R_p)^{0.25}$ Rank ordering	2.4 1.245 1.245 (1)	1.204 1.048 1.048 (2)	1 1 1 (3)
Equally weighted priority sum $0{,}25*(B_p+O_p+\frac{1}{C_p^*}+\frac{1}{R_p^*})$ Rank ordering	0.267 (3)	0.38 (1)	0.353 (2)

A summary of the findings thus far is given in Table 6. Not all supporting numerical examples are shown in this paper though.

Table 6: Summary of findings using equally weighted expressions for BOCR synthesis

	Always yields correct ordering of alternatives	Always yields correct indication of profitability
Multiplicative (expression (3))	yes (“yes” holds for any weighting scheme)	no (yes, if $B_m^t * O_m^t = C_m^t * R_m^t$)
Additive (expression (4))	no	no

4. Commensurability

The purpose of the exercise in the previous section was to show that relative priorities do not always give the best choice when choosing just the best alternative from the set, and also that the best may not be a good decision. Choosing the best alternative is a bad decision when its negatives exceed its positives. How can these phenomena be explained?
The composite priorities of the alternatives for B, O, C and R come from four separate (independent) models. On each factor, these composite priorities are in ratio form, relative to each other. For example (Table 1), A2 has 4.75 times (0.461/0.097) the overall benefit as A1 and 1.04 times (0.461/0.442) the overall benefit as A3. Notice that the sum of the benefit priorities equals one, which implies that the unit of the scale is the sum of the benefits of the three projects. The same holds for the other scales. But, although the sum of the priorities on each scale equals one, *they only represent the same if the totals of the factors are the same*.
Numbers on derived ratio scales that sum to one without the explicit specification of the unit can be deceiving. Although the individual benefits are each measured in relation to the total benefit of the projects under consideration, the priorities are only measures between alternatives and not between the types of measure. Thus, there is no assurance that the benefit priorities are commensurate with the priorities on other factors. Synthesis however requires commensurate priorities on a common scale. There is a need to know the magnitude relationship between total benefits and total costs and total

opportunities and total risks. Only in the case where these totals are all equal, a transformation into priorities normalized to the unit sum would produce commensurate priorities. As was shown earlier, also in the case where the product of benefit and opportunities would equal the product of costs and risks, the products of the corresponding normalized priorities would be in the same unit and their quotient produces meaningful BOCR ratio results.

The problem is that these totals are seldom equal to each other. It is therefore very rare that the priorities on the factors have the same unit of measure. Accordingly, an adjustment is needed to form a new common unit or to place one set of priorities in the unit of the other. The factors must be somehow linked to establish that common unit. The resulting adjusted priorities that sum to one across the four factors signify that they are in the unit of the totality of all benefits, costs, opportunities and risks. Similar to B/C analysis, BOCR synthesis of priorities is deceiving if it is composed of sets of measures that have not been adjusted relative to each other and are therefore not commensurate.

This paper argues that although any set of weights reflecting relative importance can express priorities on a common priority scale, it is only a weighting scheme based on relative B, O, C and R magnitudes that will not only serve the purpose of validation but will also allow a sound profitability analysis, equivalent to monetary break-even analysis, even when the BOCR factors are intangible.

5. Rescaling weights for commensurate priorities

Wedley et al (2001) suggested a formal magnitude adjustment procedure that converts the benefit and cost hierarchies to a common unit for B/C analysis. They suggested two different questioning methods between the two hierarchies for assessing overall benefits vs. overall costs:

1) Which perspective is more important, aggregate benefits or aggregate costs and by how many times?
2) Which perspective is more important, average alternative benefits or average alternative costs and by how many times?

In BOCR analysis opportunities and risks must also be included in the pairwise magnitude comparisons. The totals of the four factors are shown in Table 3 for each of the scenarios 1, 2 and 3. Total benefits, for example, are always twice the total opportunities in monetary units, whereas in terms of priorities they both equal unity due to the normalization to the unit sum. This implicates that the opportunity priorities should be rescaled to half of the benefit priorities.

Table 7: Synthesis using expressions (1) and (2) with magnitude-based weights that rescale composite priorities; rank ordering shown in parentheses

	Alt. A1	Alt. A2	Alt. A3
Scenario 1: w_b=s_b=0.2, w_o=s_o=0.125, w_c=s_c=0.5, w_r=s_r=0.125			
Multiplicative priority synthesis	1.215 (3)	1.31 (1)	1.285 (2)
Additive priority synthesis	0.364 (1)	0.316 (3)	0.32 (2)
Monetary ratio	0.170 (3)	0.561 (2)	0.613 (1)
Scenario 2: w_b=s_b=0.267, w_o=s_o=0.133, w_c=s_c=0.533, w_r=s_r=0.067			
Multiplicative priority synthesis	1.092 (3)	1.249 (1)	1.240 (2)
Additive priority synthesis	0.352 (1)	0.321 (3)	0.327 (2)
Monetary ratio	0.341 (3)	1.122 (2)	1.226 (1)
Scenario 3: w_b=s_b=0.331, w_o=s_o=0.165, w_c=s_c=0.496, w_r=s_r=0.008			
Multiplicative priority synthesis	0.743 (3)	1.048 (2)	1.067 (1)
Additive priority synthesis	0.310 (3)	0.338 (2)	0.352 (1)
Monetary ratio	4.541 (3)	14.964 (2)	16.341 (1)

Table 7 shows the scenario-dependent rescaling weights {s_b, s_o, s_c, s_r} computed from the total monetary values of each factor shown in Table 3, thereby reflecting the relative magnitudes of the four factors. The table also shows the results when these magnitude-based weights are used as the weights w_b, w_o, w_c and w_r in expressions (1) and (2) rather than those established by “personal values”. The composite priorities on each of the four factors are again those from Table 1.

However, despite use of rescaled composite priorities when multiplicatively or additively synthesizing, correct final rank ordering of the alternatives or correct profitability indications are still not always obtained!

The additional findings are summarized in Table 8. Not all supporting numerical examples are shown in this paper though.

Table 8: Summary of findings using expressions with rescaling weights for BOCR synthesis

	Always yields correct ordering of alternatives	Always yields correct indication of profitability
Multiplicative (expression (1))	no	no
Additive (expression (2))	no	no

The conclusion from Tables 6 and 8 would be that there is no guarantee, except in very special (and probably rare) cases, that either the multiplicative or the additive expression suggested thus far in BOCR analysis yields meaningful results, even if weights are used that rescale composite priorities from different BOCR scales in accordance with the relative magnitudes of the four factors.
The next section will show that the reason for this is that a) the rescaling weights appeared as powers in the multiplicative synthesis expression and not as coefficients, and b) the rescaling weights related the magnitudes per se instead of the form in which they appear in the synthesis expression.

6. Revised expressions with commensurate priorities

In this section some revised expressions for BOCR synthesis using commensurate priorities are proposed that more closely reflect monetary BOCR analysis.
When one would know monetary values for each of the four factors, and would consider one benefit, one opportunity, one cost and one risk dollar all equally important, the following simple expression would most likely be used for the *i*-th alternative:

$$\frac{B_m^i * O_m^i}{C_m^i * R_m^i} \tag{6}$$

The equivalent expression with rescaled and therefore commensurate priorities is:

$$\frac{(s_b * B_p^i) * (s_o * O_p^i)}{(s_c * C_p^i) * (s_r * R_p^i)} \tag{7}$$

where $\{s_b, s_o, s_c, s_r\}$ is the set of weights that rescale the priorities, as coefficients, not powers:

$$s_b : s_o : s_c : s_r = B_m^t : O_m^t : C_m^t : R_m^t \tag{8}$$

The proof that (6) and (7) are equivalent under this condition is straightforward and will not be shown.

There is, however, no obvious reason why one would compute the product of benefits and opportunities or the product of costs and risks, sums may be equally useful but with a different meaning. Expressions (6) and (7) would then have to be slightly changed into a quotient of *sums* rather than products of rescaled priorities. Both perspectives allow a break-even analysis based on the unity value indicating the break-even point, be it with a different interpretation. Table 9 contains results showing perfect validation for the three scenarios if commensurate benefits and opportunities are *added* and the same with costs and risks.

Other expressions might be designed as well. There will however only exist a full equivalency between the outcome of monetary computations (remember, for validation purposes) and that of priority-based computations *if the same type of expression for synthesis is used* and if and only if the priorities are rescaled (and thus made commensurate) so as to represent the relative magnitudes of the four factors BOCR *according to the way in which they appear in the formula*.
The additive synthesis proposed by Saaty (2004) would therefore have to be changed into:

$$s_b^{'} * B_p^i + s_o^{'} * O_p^i + s_c^{'} * \frac{1}{C_p^i} + s_r^{'} * \frac{1}{R_p^i} \tag{9}$$

where the rescaling weights, denoted by $\{s_b^{'}, s_o^{'}, s_c^{'}, s_r^{'}\}$, are now proportionate to the totals of benefits and opportunities and *to the reciprocals of the totals of costs and risks*! The priority-based outcome of (9) would have to validated against the monetary outcome of (10):

$$B_m^i + O_m^i + {}^{1}\!/_{C_m^i} + {}^{1}\!/_{R_m^i} \qquad (10)$$

Additive synthesis still keeps the disadvantage that no break-even-based profitability indication is offered.

Table 9 summarizes the scenario-dependent results for each of the expressions above, comparing monetary results with the priority-based results. Although not all examples are shown in this paper, both types of result do always fully coincide.

Table 9: Synthesis with different expressions using magnitude-based weights that rescale composite priorities

	Alt. A1	Alt. A2	Alt. A3
Scenario 1: s_b=0.25, s_o=0.125, s_c=0.5, s_r=0.125			
Monetary product quotient (6)	0.170	0.561	0.613
Priorities-based product quotient (7)	0.170	0.561	0.613
Monetary sum quotient	0.329	0.659	0.664
Priorities-based sum quotient	0.329	0.659	0.664
Monetary additive (10) (+ normalized) *)	612 (0.102)	2556 (0.426)	2832 (0.472)
Priorities-based additive (9)	0.102	0.426	0.472
Scenario 2: s_b=0.267, s_o=0.133, s_c=0.533, s_r=0.067			
Monetary product quotient (6)	0.341	1.122	1.226
Priorities-based product quotient (7)	0.341	1.222	1.226
Monetary sum quotient	0.361	0.73	0.745
Priorities-based sum quotient	0.361	0.73	0.745
Monetary additive (10) (+ normalized) *)	612 (0.102)	2556 (0.426)	2832 (0.472)
Priorities-based additive (9)	0.102	0.426	0.472
Scenario 3: s_b=0.331, s_o=0.165, s_c=0.496, s_r=0.008			
Monetary product quotient (6)	4.541	14.964	16.341
Priorities-based product quotient (7)	4.541	14.964	16.341
Monetary sum quotient	0.526	1.072	1.109
Priorities-based sum quotient	0.526	1.072	1.109
Monetary additive (10) (+ normalized) *)	612 (0.102)	2556 (0.426)	2832 (0.472)
Priorities-based additive (9)	0.102		0.472

**) results from expressions (9) and (10) seem identical across the scenarios but are actually negligibly different*

One may wonder if there is still a possibility to use weights based on personal values, as suggested by Saaty (2001). The answer is "yes", but then not the composite priorities themselves must be weighted by them but rather the *rescaled* (commensurate) composite priorities. The indication of profitability which enables a break-even analysis based on the unity value will, however, be affected accordingly. For fair validation purposes however, monetary values will also have to be weighted by those "personal" weights.

7. Concluding remarks

In this paper benefits-opportunities-costs-risks (BOCR) analysis using AHP/ANP methodology was addressed. The computation of the composite priorities for the alternatives on each of the four BOCR factors may be based on hierarchy (AHP) or network (ANP) models, but is not the subject of this paper. It is rather the final synthesis of these four types of composite priorities, using a multiplicative or an additive expression as advocated in the AHP/ANP literature, that was investigated. In order to be able to draw conclusions as to the validity of these two types of synthesis, monetary equivalents of priorities were used in several scenarios. In reality, these monetary values are not known; they were merely used to investigate whether or not synthesis of priorities reproduces monetary results, assuming perfect consistency.

The analysis in this paper suggests that it is crucial to express priorities on benefits, opportunities, costs and risks in commensurate terms. Otherwise, the results are meaningless or even deceiving (by incorrectly suggesting profitability of alternatives) and contradictory (showing rank reversals with results from more or less equivalent synthesis expressions), leading to bad decisions. This paper argues that although any set of weights reflecting relative importance can express priorities on a common priority scale, it is only a weighting scheme based on relative B, O, C and R magnitudes that will not only serve the purpose of validation but will also allow a sound profitability analysis, equivalent to monetary break-even analysis, even when the BOCR factors are intangible.

This paper also argues that a revised synthesis expression should be used that is not additive or a quotient of power functions but rather a quotient of positives (B and O) to negatives (C and R) with the rescaling weights as coefficients, not powers. The revised synthesis expression comes in two variants: one where rescaled (and thus commensurate) benefit and opportunity priorities are added in the nominator (and the same with the rescaled cost and risk priorities in the denominator), and one where they are multiplied. Furthermore, the mathematical function for synthesis of priorities should be equivalent to the function that would be used if objective (e.g. monetary) values were known, in order to get close (or, ideally, perfect) approximations to known outcomes. Finally, if additional weights based on personal values are to be used to take account of feelings of relative importance of the four factors, this paper suggests that they should be applied to the rescaled priorities.

Similar questions as the ones proposed by Wedley et al. (2001) enable BOCR priorities to achieve the desirable attributes of regular BOCR analysis. The disadvantage however is the cognitive difficulty to compare aggregate (or average) BOCR factors. Potentially simpler techniques for achieving commensurability were proposed by Wedley et. al. (2003) using linking pin technology. Since the unit of a derived ratio scale is arbitrary, a proportional transformation of the scale can put the unit in any hierarchy node. Furthermore, the link between hierarchies does not have to be at their topmost node (Schoner et al.; 1993) but across a common alternative or well-chosen specific sub-factors. Before linking, one merely has to identify a node that becomes the unit of measurement for each hierarchy. Other nodes are then expressed in terms of that unit. Comparing abstract totalities seems mentally more demanding than comparing single nodes. However, how this should be done when having networks for the BOCR factors rather than hierarchies remains open for further investigation.

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